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Chapter 1

Introduction générale

Ce manuscrit présente mes travaux de recherche, faits de Septembre 2015 à Mai 2018, au cours de ma thèse sous la direction de Hans Henrik Rugh.
Le thème principal porte sur le développement de théorèmes perturbatifs en théorie ergodique des systèmes hyperboliques. En d'autres termes, il s'agit de comprendre comment les propriétés statistiques d'une dynamique présentant des aspects chaotiques évoluent sous l'influence de perturbations extérieures.

Ces systèmes chaotiques se caractérisent par la présence de la propriété de *sensibilité aux conditions initiales*

$$\exists D > 0,\ \forall\ x \neq y \in X^2, \exists N = N(x, y),\ d(T^N x, T^N y) \geq D \tag{1.0.1}$$

En d'autres termes, deux orbites issues de points distincts s'éloignent exponentiellement vite l'une de l'autre, jusqu'à un certain point. Cela rend peu pertinente l'étude topologique des orbites individuelles et légitime une approche probabiliste de l'étude de tels systèmes. Une telle étude probabiliste repose sur la méthode fonctionnelle, c'est-à-dire les propriétés spectrales de l'opérateur de transfert (2.2.7), (2.2.11) et les liens qu'elles entretiennent avec le comportement ergodique du système : construction de mesures physiques et d'états d'équilibres, étude des fonctions de corrélations et vitesse de mélange, théorèmes limites...
Nous faisons quelques rappels sur la méthode fonctionnelle en §2.2.

Les travaux ici présentés portent essentiellement sur les applications présentant la propriété de dilatation, c'est-à-dire augmentant les petites distances. Plus précisément, il s'agit d'applications de classe C^r, $r > 1$ tel qu'il existe $\lambda > 1$ vérifiant

$$\forall x \in M,\ \forall v \in T_x M, \|DT(x).v\| \geq \lambda \|v\| \tag{2.1.4}$$

Ces systèmes sont l'exemple le plus simple de dynamique chaotique au sens de (1.0.1), avec une structure d'orbite véritablement complexe et des phénomènes de récurrence non triviaux [48, §1.7, p.41].

Au nombre des systèmes dilatants présentant une "pertinence physique" (c'est-à-dire étudié par d'autres que les mathématiciens), on peut citer les modèles de turbulence de Pommeau-Manneville ou encore les *sous-décalages de type fini*, initialement issus d'un modèle de physique statistique de spins à l'équilibre. Pour une discussion plus poussée des propriétés de tels systèmes, ainsi que des exemples, nous renvoyons aux §2.1 et §2.4.

Au registre des critiques que l'on peut légitimement adresser à ce choix, on peut alléguer du peu de pertinence physique d'un modèle ne faisant intervenir que des applications dilatantes : bien souvent les modèles issus de la physique, de la chimie, de la climatologie ou autres font appel à des applications "Axiom A" ou partiellement hyperboliques.

Toutefois, le choix d'étudier des modèles dilatants plutôt que ceux-ci se justifie par la présence de propriétés chaotiques similaires à celle de modèles plus complexes, mais dans un contexte techniquement plus simple où les idées-force se comprennent d'autant plus facilement. Par exemple, l'opérateur de transfert d'une application dilatante a de "bonnes" propriétés spectrales sur les espaces fonctionnels classiques (Sobolev [6, 75] ou Hölder [63, 64, 41]), alors que pour voir de telles propriétés émerger pour des systèmes Anosov, il faut travailler sur des espaces de distributions anisotropes ad-hoc, dont la construction est beaucoup plus technique et l'apparition nettement plus récente ([39, 40] [8]).
Par ailleurs, la semi-conjugaison entre systèmes Anosov et sous-décalage de type fini, au cœur du formalisme thermodynamique, justifie qu'historiquement on se soit intéressé aux systèmes dilatants.

Essayons de formaliser notre propos : étant donné $r > 1$ et $(-a, a) \subset \mathbb{R}$ un intervalle ouvert, nous considérons une famille $(T_\epsilon)_{\epsilon\in(-a,a)}$ d'applications C^r sur une variété Riemannienne M, et telle que

- Pour chaque $\epsilon \in (-a, a)$, T_ϵ admet une mesure physique ν_ϵ(cf. §2.2 pour une définition)
- L'application $\epsilon \in (-a, a) \mapsto T_\epsilon \in C^r(M)$ est de classe C^s pour un $s \geq 1$.

Nous dirons que le système a la propriété de **réponse linéaire** si $\epsilon \in (-a, a) \mapsto \nu_\epsilon$ est C^1 en $\epsilon = 0$ au sens des distributions, i.e s'il existe une mesure μ telle que pour toute $\phi \in C^0(M)$

$$\frac{1}{\epsilon}\left[\int_M \phi d\nu_\epsilon - \int_M \phi d\nu_0\right] - \int_M \phi d\mu \underset{\epsilon\to 0}{\longrightarrow} 0 \tag{1.0.2}$$

Illustrons tout de suite les notions précédentes en présentant un résultat de réponse linéaire dans le cas le plus favorable, celui d'une perturbation analytique (i.e $s = \omega$) d'une application uniformément dilatante du cercle :

Theorem 1.1
Soit $a > 0$, et pour $\epsilon \in (-a, a)$, on considère $T_\epsilon \in \mathcal{A}(\mathbb{S}^1, \mathbb{S}^1)$, définie par

$$T_\epsilon(x) = 2x + \epsilon \sin(2\pi x) \mod 1 \tag{1.0.3}$$

Alors si on note $\mathcal{L}_\epsilon$ l'opérateur de transfert associé à T_ϵ, on a (quitte à réduire $a > 0$)

(i) Pour chaque $\epsilon \in (-a, a)$, T_ϵ admet une mesure de probabilité invariante absolument continue par rapport à la mesure de Lebesgue, de densité $h_\epsilon \in \mathcal{A}(\mathbb{S}^1)$.

(ii) L'application $\epsilon \in (-a, a) \mapsto h_\epsilon \in \mathcal{A}(\mathbb{S}^1)$ est analytique, et on a la formule suivante pour sa dérivée en 0 :

$$[\partial_\epsilon h_\epsilon]_{\epsilon=0} = (\mathbb{1} - \mathcal{L}_0)^{-1}[\partial_\epsilon \mathcal{L}_\epsilon]_{\epsilon=0} h_0 \tag{1.0.4}$$

Ce résultat fait partie du folklore : on en retrouve la trace dans [4, Exercice 2.25, p.135], ou encore dans [61] par une méthode différente. Sa preuve repose sur les remarques suivantes

- L'existence de la densité de la mesure physique h_ϵ découle de la théorie classique des perturbations (plus précisément de [47, IV, §3, Thm 3.12 et VII, § Thm 1.8]), car c'est un point fixe de l'opérateur de transfert $\mathcal{L}_\epsilon$.
- $\epsilon \in (-a, a) \mapsto \mathcal{L}_\epsilon \in L(\mathcal{A}(\mathbb{S}^1))$ étant analytique, le second point provient du théorème des fonctions implicites pour applications analytiques sur des espaces de Banach.

Ce second point garantit que dans le cas décrit par le théorème, il y a bien réponse linéaire, et même régularité en un sens plus fort, puisque c'est la densité de la mesure invariante qui est analytique par rapport aux paramètres.

La formule (1.0.4), dite *formule de réponse linéaire*, mérite quelques explications : le terme de droite représente le changement quantitatif de la mesure physique lorsque la dynamique passe de T_0 à T_ϵ. Il fait intervenir la résolvante de l'opérateur de transfert en 1, a priori mal définie puisque celui-ci admet $h_0 = 1$ pour point fixe. Toutefois, en choisissant une bonne normalisation pour h_ϵ on peut s'assurer que $[\partial_\epsilon \mathcal{L}_\epsilon]_{\epsilon=0} h_0$ se trouve dans un sous-espace où cette résolvante est bien définie (cf §2.4).

Malheureusement, dès que l'on sort du cadre analytique, l'opérateur de transfert, si naturel pour décrire les propriétés statistiques du système, a une limitation intrinsèque liée au fait qu'il soit un opérateur de composition : cette limitation, c'est la *perte de régularité*, c'est-à-dire l'absence de régularité par rapport aux paramètres dans la topologie d'opérateur.

Pour illustrer notre propos, considérons l'exemple suivant : soit $0 < a \leq 1/4$ et $\mathcal{M}_\epsilon$ l'opérateur de composition défini pour $\phi \in C^1([-1, 1])$ et $\epsilon \in [-a, a]$ par

$$\mathcal{M}_\epsilon \phi(x) := \phi\left(\epsilon + \frac{x}{2}\right) \tag{1.0.5}$$

Il apparaît que si chaque $\mathcal{M}_\epsilon$ est bien borné sur $C^1([-1, 1])$, l'application $\epsilon \in [-a, a] \mapsto \mathcal{M}_\epsilon \in L(C^1([-1, 1]))$ n'est même pas continue... En revanche, l'application

$$\epsilon \in [-a, a] \mapsto \mathcal{M}_\epsilon \in L(C^1([-1, 1]), C^0([-1, 1])$$

est continue, et même C^1.

Ce phénomène se retrouve au niveau de l'opérateur de transfert lorsqu'on considère des perturbations en régularité finie. En effet, reprenons l'exemple du 1.1, modifié de la façon suivante : au lieu de considérer une perturbation analytique de la dynamique T_0, considérons une perturbation C^1 (i.e telle que $\epsilon \in (-a, a) \mapsto T_\epsilon \in C^3(\mathbb{S}^1, \mathbb{S}^1)$ soit C^1), suffisamment petite pour que T_ϵ soit également une application uniformément dilatante de degré 2, de branches inverses $\psi_1(\epsilon)$ et $\psi_2(\epsilon)$. Par exemple, on peut prendre la fonction induite sur le cercle $\mathbb{S}^1$ par

$$\begin{cases} u(x) = \sin(2\pi x) \\ T_\epsilon(x) = 2x + \epsilon u(x)^5 \sin\left(\dfrac{1}{u(x)}\right) \mod 1 \end{cases} \tag{1.0.6}$$

L'opérateur de transfert de T_ϵ s'écrit alors

$$\mathcal{L}_\epsilon\phi(x) = \frac{\phi(\psi_1(\epsilon,x))}{T'_\epsilon(\psi_1(\epsilon,x))} + \frac{\phi(\psi_2(\epsilon,x))}{T'_\epsilon(\psi_2(\epsilon,x))}$$

Comme dans l'exemple (1.0.5), l'application $\epsilon \in (-a,a) \mapsto \mathcal{L}_\epsilon \in L(C^2(\mathbb{S}^1))$ n'est pas continue.

La limitation apparaissant dans les exemples précédents sont typiques du genre de problème technique que l'on rencontre lors de l'étude de problèmes perturbatifs formulés dans le cadre fonctionnel. Nous verrons comment surmonter ce problème dans le chapitre 3, en adaptant l'idée qui se dégage de l'exemple (1.0.5) : échanger la régularité de l'espace fonctionnel sur lequel l'opérateur agit contre de la régularité par rapport aux paramètres.

Plus précisément, après avoir introduit la notion d'application différentiable graduée (3.1), nous montrons le théorème 3.1. Ce résultat est en un sens la base technique des travaux présentés ici, puisqu'il permet une étude systématique des points fixes d'applications ayant une perte de régularité. Nous présentons également une généralisation, le théorème 3.2, afin d'étudier les cas de différentiabilité d'ordre plus grand que 1.

Cet apport théorique nous permet de présenter au chapitre 4 une étude de la régularité par rapport aux perturbations des propriétés statstiques des applications C^r, $r > 1$, uniformément dilatante sur une variété Riemannienne compacte, connexe de dimension finie.
En particulier, nous pouvons montrer le théorème suivant :

Theorem 1.2
Soit $T_\epsilon \in C^3(\mathbb{S}^1, \mathbb{S}^1)$ définie par (1.0.6). Alors on a (quitte à réduire $a > 0$):

- *Pour chaque $\epsilon \in (-a,a)$, T_ϵ admet une mesure de probabilité invariante et absolument continue, de densité $h_\epsilon \in C^2(\mathbb{S}^1)$.*
- *L'application $\epsilon \in (-a,a) \mapsto h_\epsilon \in C^1(\mathbb{S}^1)$ est dérivable, et on a la formule suivante pour sa dérivée en 0*
$$[\partial_\epsilon h_\epsilon]_{\epsilon=0} = (\mathbb{1} - \mathcal{L}_0)^{-1}[\partial_\epsilon \mathcal{L}_\epsilon]_{\epsilon=0} h_0 \tag{4.1.2}$$

Encore une fois, le second point du théorème est une version forte de réponse linéaire. Par rapport au théorème 1.1, notons que, comme annoncé, il faut voir la densité invariante comme une fonction moins régulière que ce qu'elle n'est (C^1 au lieu de C^2) pour gagner la régularité C^1 par rapport aux paramètres.

Le théorème 1.2 fait encore une fois partie du folklore : on le retrouve dans [32, Lemma 1.2] ou dans [2, Theorem 2.2]. Nous présentons en §2.5 une brève recension des méthodes de preuve de 1.2 : méthode "pédestre" de [2], ou la méthode plus subtile de Gouëzel-Liverani [39, Theorem 8.1], dite de *perturbation spectrale faible*, et qui a le mérite de se généraliser à toutes les situations où la régularité et l'hyperbolicité sont uniformes (voir par exemple [6, §2.5 et §5.3] ou les articles originaux [39, 40]).

Plus généralement, les résultats présentés au chapitre 4 ne sont pas spécialement nouveaux. Là où notre approche donne des résultats véritablement inédits concerne les produits aléatoires d'applications dilatantes (chapitre 5),

$$T_\omega^{(n)} := T_{\tau^{n-1}\omega} \dots T_\omega \tag{1.0.7}$$

dirigés par un système dynamique mesurable $(\Omega, \tau, \mathbb{P})$. Ce formalisme permet l'étude de systèmes dynamique non-autonomes. Dans ce cadre l'objet dynamiquement pertinent devient le *cocycle d'opérateurs de transferts*

$$\mathcal{L}_\omega^{(n)} := \mathcal{L}_{\tau^{n-1}\omega} \dots \mathcal{L}_\omega$$

et plus spécifiquement son spectre d'Oseledets-Lyapunov [29, 18], qui joue un rôle similaire à celui du spectre de l'opérateur de transfert dans l'étude des propriétés statistiques d'un système dynamique autonome.
En particulier il devient impossible d'utiliser l'approche spectrale de Gouëzel-Keller-Liverani pour étudier les problèmes de perturbations des mesures (aléatoires) invariantes d'un tel système non-autonome.

En combinant notre approche "fonctions implicites" (théorèmes 3.1, 3.2) avec la théorie des contractions de cônes (pour laquelle nous donnons des rappels dans l'appendice A), nous construisons l'unique mesure stationnaire absolument continue d'un produit aléatoire (1.0.7), et étudions la régularité de sa densité aléatoire par rapport à un nombre fini de paramètres (théorèmes 5.1, 5.2). Nous établissons également une formule de réponse linéaire dans ce contexte, à la fois *quenched* (5.2.27 dans le théorème 5.4) et *annealed* (5.2.39 dans le théorème 5.5) : c'est à la connaissance de l'auteur la première fois que la réponse linéaire est établie pour des systèmes dynamiques non-autonomes.

Voici un exemple de théorème que nous pouvons établir avec les outils du chapitre 5 : on construit un produit aléatoire d'applications dilatantes du cercle de la manière suivante. Si l'on note

$$\begin{cases} u(x) = \sin(2\pi x) \\ T_\epsilon(x) = 2x + \epsilon u(x)^5 \sin\left(\dfrac{1}{u(x)}\right) \mod 1 \\ S_\epsilon(x) = 3x + \epsilon u(x)^5 \sin\left(\dfrac{1}{u(x)}\right) \mod 1 \end{cases} \tag{1.0.8}$$

et que l'on considère de décalage de Bernoulli τ sur $\Omega = \{0,1\}^{\mathbb{Z}}$, muni de la mesure de Bernoulli[1] $((1-p)\delta_0 + p\delta_1)^{\otimes \mathbb{Z}}$, on peut définir une application dilatante aléatoire $T_{\omega,\epsilon}$, $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ sur le cercle $\mathbb{S}^1$ par

$$T_{\omega,\epsilon}(x) := \begin{cases} T_\epsilon(x) \text{ si } \omega_0 = 0 \\ S_\epsilon(x) \text{ si } \omega_0 = 1 \end{cases} \tag{1.0.9}$$

On a alors le résultat suivant :

Theorem 1.3
Considérons le produit aléatoire engendré par (1.0.9)

$$T_{\omega,\epsilon}^{(n)} := T_{\tau^{n-1}\omega,\epsilon} \dots T_{\omega,\epsilon} \tag{1.0.10}$$

Il existe $a > 0$ *tel que :*

[1] c'est-à-dire la mesure de Markov associée au couple

$$\left(\begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}, (p, 1-p) \right)$$

Plus généralement on peut prendre la mesure de Markov associé à n'importe quelle matrice stochastique P, l'important étant que le système résultant soit ergodique.

- *Pour chaque* $\epsilon \in (-a,a)$*, le* skew-product $F_\epsilon(\omega,x) := (\tau\omega, T_{\omega,\epsilon}(x))$ *admet une unique probabilité invariante, absolument continue, de densité aléatoire* $h_{\omega,\epsilon} \in C^2(\mathbb{S}^1)$*.*
  De plus, $\operatorname{ess\,sup}_{\omega\in\Omega} \|h_{\omega,\epsilon}\|_{C^2(\mathbb{S}^1)} < +\infty$*.*
- *L'application* $\epsilon \in (-a,a) \mapsto h_\epsilon \in L^\infty(\Omega, C^1(\mathbb{S}^1))$ *est différentiable, et on a les formules de réponse linéaire suivantes :*

$$\partial_\epsilon \left[\int_M \phi h_{\omega,\epsilon} dm\right]_{\epsilon=0} = \sum_{n=0}^{\infty} \int_M \phi \circ T^{(n)}_{\tau^{-n}\omega,0} \partial_\epsilon \mathcal{L}_{\tau^{-(n+1)}\omega,0} h_{\tau^{-(n+1)}\omega,0} dm \tag{1.0.11}$$

$$\partial_\epsilon \left[\int_\Omega \int_M \phi h_{\omega,\epsilon} dm d\mathbb{P}\right]_{\epsilon=0} = \sum_{n=0}^{\infty} \int_\Omega \int_M \phi \circ T^{(n)}_{\omega,0} \partial_\epsilon \mathcal{L}_{\tau^{-1}\omega,0} h_{\tau^{-1}\omega,0} dm d\mathbb{P} \tag{1.0.12}$$

Nous établissons également des résultats de régularité pour diverses quantités dynamiques d'intérêt : régularité de la variance dans le théorème limite central vérifié par un produit aléatoires d'applications dilatantes du tore (theorème 5.3), régularité de la dimension de Hausdorff pour le *repeller* associé à un produit aléatoire de cookies-cutters (§ 5.2.3).

Chapter 2

Préliminaires

2.1 Quelques propriétés des systèmes dilatants

Soit (M, d) un espace métrique compact, connexe. Nous définissons une application (uniformément) dilatante continue comme une $T : M \to M$ continue, telle qu'il existe $\lambda > 1$ et $b > 0$ pour lesquels

$$d(T(x), T(x')) \geq \lambda d(x, x') \text{ dès que } d(x, x') < b \tag{2.1.1}$$

Il est clair qu'un telle application satisfait (1.0.1). Donnons quelques exemples de systèmes satisfaisant cette propriété :

- L'application de doublement de l'angle, définie sur le cercle $\mathbb{S}^1$ par

$$T(x) = 2x \mod 1$$

- Plus généralement, toute $A \in \mathcal{M}_d(\mathbb{Z})$, telle que $\sigma(A) \subset \{z, |z| > 1\}$ induit une application dilatante $\widetilde{A} : \mathbb{T}^d \to \mathbb{T}^d$.

- Toute $T : [0, 1] \to [0, 1]$ de classe C^r, $r > 1$, telle que $\inf_{[0,1]} |T'| > 1$ est une application dilatante.

- Soit $I_1,\ I_2 \subset [0, 1]$ deux intervalles disjoints, et soit $T : I_1 \cup I_2 \to [0, 1]$ de classe C^2 telle que :

  - $T_{|I_i}$ est affine pour $i = 0, 1$
  - $T(I_i) = [0, 1]$ pour $i = 0, 1$
  - $\inf_{I_i} |T'| > 1$

  Alors T est une application uniformément dilatante.

- Soit I un ensemble fini, et soit π une matrice de taille $|I| \times |I|$, telle que $\pi_{i,j} \in \{0, 1\}$ pour tout $(i, j) \in I^2$. Considérons l'ensemble des *suites admissibles*

$$\Sigma_I := \{x = (x_n)_{n\in\mathbb{N}},\ x_n \in I,\ \pi_{x_n,x_{n+1}} = 1\}$$

et $\tau : \Sigma_I \to \Sigma_I$ le *sous-décalage* associé, défini par

$$(\tau(x))_n = (x_{n+1})$$

Si $x = (x_n)_{n\geq 0} \in I^{\mathbb{N}}$ et $y = (y_n)_{n\geq 0} \in I^{\mathbb{N}}$, on note $k(x,y) := \min\{n \geq 0,\ x_n \neq y_n\}$, et pour $\theta \in (0,1)$ on définit la distance d_θ sur $I^{\mathbb{N}}$ par

$$d_\theta(x,y) = \theta^{k(x,y)}$$

Alors (Σ_I, d) est une partie compacte de $(I^{\mathbb{N}}, d)$, et le sous-décalage $\tau : \Sigma_I \to \Sigma_I$ est continu et uniformément dilatant.

Certains systèmes vérifient une propriété très proche de (2.1.1), les applications **non-uniformément dilatantes**. Elles vérifient (2.1.1), sauf en un point.

Un exemple important est le suivant : considérons la famille d'applications $(T_\alpha)_{\alpha\in(0,1)}$, dites intermittentes, définie sur $[0,1]$ par

$$T_\alpha(x) := \begin{cases} x(1+2^\alpha x^\alpha) \text{ si } 0 \leq x \leq 1/2 \\ 2x-1 \text{ si } 1/2 \leq x \leq 1 \end{cases} \tag{2.1.2}$$

Chaque T_α admet une discontinuité en $1/2$, envoie chacun des deux intervalles $[0,1/2]$ et $(1/2,1]$ sur $[0,1]$ tout entier, a une branche de droite affine et uniformément dilatante (de dérivée plus grande que 2), mais la branche de gauche admet en 0 un point fixe neutre (i.e $T'_\alpha(0) = 1$) : ainsi T_α n'est pas uniformément dilatante.

Nous allons dans cette section établir quelques unes des propriétés topologiques et géométriques élémentaires de la classe des systèmes uniformément dilatants. De bonnes références sur le sujet sont la recension de Fan et Jiang [23], ainsi que le papier fondateur de Shub [74].

Les propositions et preuves qui suivent sont tirées de [23, §2].

Commençons par remarquer qu'une application $T : M \to M$ satisfaisant 2.1.1 est un homéomorphisme local; en particulier, c'est une application ouverte.

Proposition 2.1
Soit $T : M \to M$ une application continue uniformément dilatante. Alors T est un homéomorphisme local.

En effet, on a que $T : B(x,b) \to T(B(x,b))$ est un homéomorphisme : l'injectivité découle directement de (2.1.1), la surjectivité est évidente, et comme c'est une bijection continue sur le compact $B(x,b)$, il en va de même de sa réciproque, ce qui nous donne le résultat. □

On peut alors s'interroger sur les propriétés des branches inverses de T:

Proposition 2.2
Soit $T : M \to M$ une application continue et uniformément dilatante.

- *Chaque $y \in M$ admet un nombre fini d'antécédents par T, et mieux encore, lorsque M est connexe il existe $p < +\infty$ tel que $\#T^{-1}(y) = p$.*

- *Il existe $a > 0$ tel qu'en chaque $y \in M$, T admette un nombre fini de branches inverses $(\psi_i)_{i=1\ldots p}$, et les $(\psi_i(B(y,a)))_{i=1\ldots p}$ soient 2 à 2 disjointes.*

- *Si* $0 < r \leq a$*, et que* ψ *est une branche inverse de* T *définie sur* $B(y,r)$*, alors pour tout* $z, z' \in B(y,r)$*, on a*
$$d(\psi(z), \psi(z')) \leq \frac{d(z,z')}{\lambda}$$

Soit $y \in M$. On commence par remarquer que $T^{-1}(y)$ est un ensemble fini, puisque discret dans un compact: si tel n'était pas le cas, il existerait $x \in T^{-1}(y)$, tel que $B(x,r) \cap T^{-1}(y) \neq \emptyset$ pour tout $r > 0$: cela contredit le caractère d'homéomorphisme local de T. On a donc $\#T^{-1}(y) = p(y) < +\infty$. Considérons donc $x_1, \ldots, x_{p(y)}$ les antécédents de y par T. Si on note $d_y = \min_{i,j=1,\ldots,p(y)} d(x_i, x_j)$, il est clair que $d_y > b$.
Par la proposition précédente, pour tout $0 < r \leq \frac{b}{2}$, $T : B(x_i, r) \to T(B(x_i, r))$ est un homéomorphisme, pour tout $i \in \{1, \ldots, p(y)\}$. Notons $\psi_{i,y} : T(B(x_i, r)) \to B(x_i, r)$ les homéomorphismes inverses.

Notons que $y \in \bigcap_{i=1}^{p(y)} T(B(x_i, r))$, intersection finie d'ouverts, donc il existe $r_y > 0$ tel que $\overline{B(y, r_y)} \subset \bigcap_{i=1}^{p(y)} T(B(x_i, r))$.
En particulier, les $\psi_{i,y} : \overline{B(y, r_y)} \to B(x_i, r)$ sont tels que les $\psi_{i,y}(\overline{B(y, r_y)})$ sont deux à deux disjoints.

On considère désormais un nombre fini de boules de la forme $B(y, r'_y)$, $(B(y_i, r'_i))_{i=1\ldots p}$ tel que les $B(y_i, r'_i/2)$ forment un recouvrement (fini) de M.
Si on note $a = \min r'_i/2$, alors les $B(y_i, a)$ vérifient la propriété voulue. En effet, pour tout $y \in M$, on a $y \in B(y_i, r'_i/2)$ pour un certain $i \in \{1, \ldots, p\}$, et donc $\overline{B(y,a)} \subset \overline{B(y_i, r'_i)}$. Ainsi les

$$\psi_i = \psi_{i,y_i} | B(y,a), \ i \in \{1, \ldots, n\}$$

sont les branches inverses de T en y, et vérifient bien la propriété voulue.
Enfin, on peut remarquer que $\#T^{-1}(y) \leq p$ est localement constant, donc constant puisque M est connexe. $\square$

Soit $x \in M$, et soit $0 < r < a$ où a est la constante donnée par la proposition précédente. Alors pour toute branche inverse ψ définie sur $B(T(x), r)$, pour tout $w, z \in B(T(x), r)$, on a

$$d(\psi(w), \psi(z)) \leq \frac{1}{\lambda} d(w,z) \tag{2.1.3}$$

Cette propriété est parfois appelée pistage fort (ou *strong shadowing* en anglais), signifiant que l'on dispose d'un contrôle fin sur les trajectoires passées de deux points proches (à défaut de contrôler la distance entre les orbites futures)

Introduisons maintenant la distance de Bowen

$$d_n(x,y) := \max_{0 \leq i \leq n} d(f^i(x, f^i(y)))$$

ainsi que la boule de rayon r pour cette distance,

$$B_n(x,r) := \{y \in M, d_n(x,y) \leq r\} = B(x,r) \cap \cdots \cap T^{-n}\left(B(T^n(x), r)\right)$$

Il est clair que $T : B_1(x,r) \to B(T(x), r)$ est un homéomorphisme pour $r < a$. C'est encore vrai pour les itérées de T:

Proposition 2.3
Pour tout $0 < r < a$, $T^n : B_n(x,r) \to B(T^n(x),r)$ *est un homéomorphisme.*

Il est clair que $T^n : B_n(x,r) \to B(T^n(x),r)$ est injective. Vu la continuité de T^n sur le compact $B_n(x,r)$, il suffit de montrer qu'elle est surjective pour conclure que c'est un homéomorphisme.

Considérons $x \in M$, et soit $x, T(x), \dots, T^n(x)$ les n premiers points de l'orbite de x. On sait qu'il existe des branches inverses $\psi_1, \dots, \psi_n$ telles que

$$x \xleftarrow{\psi_1} T(x) \xleftarrow{\psi_2} \dots \xleftarrow{\psi_n} T^n(x)$$

Par la remarque précédant l'énoncé de la proposition, on a que $\psi_n : B(T^n(x),r) \to B_1(T^{n-1}(x),r)$ est un homéomorphisme, dès que $0 < r \leq a$.
Supposons que $\psi_{n-k-1} \circ \cdots \circ \psi_n(B(T^n(x),r)) = B_k(T^{n-k}(x),r)$. Alors il s'ensuit que $\psi_{n-k} \circ \cdots \circ \psi_n(B(T^n(x),r)) = \psi_{n-k}(B_k(T^{n-k}(x),r))$.
Or, $z \in \psi_{n-k}(B_k(T^{n-k}(x),r)) \Leftrightarrow T(z) \in B_k(T^{n-k}(x),r)$ ce qui équivaut à

$$\forall i \in \{0,\dots,k\},\ d(T^{i+1}(z), T^{n-k+i}(x)) < r$$

Mais comme $d(z, T^{n-k-1}(x)) \leq \dfrac{d(T(z), T^{n-k}(x))}{\lambda} < \dfrac{r}{\lambda}$, on a également $z \in B(T^{n-k-1}(x),r)$, d'où l'on déduit que $\psi_{n-k}(B_k(T^{n-k}(x),r)) = B_{k+1}(T^{n-k-1}(x),r)$. Ainsi, par récurrence on a montré que

$$\psi_1 \circ \cdots \circ \psi_n(B(T^n(x),r)) = B_n(x,r)$$

d'où la surjectivité de $T^n : B_n(x,r) \to B(T^n(x),r)$. □

Étant donné (M,d) un espace métrique compact, nous rappelons les définitions suivantes :

Definition 2.1

- *Une application continue* $T : M \to M$ *est dite* topologiquement transitive *s'il existe une orbite dense pour* T.
- *Une application continue* $T : M \to M$ *est dite* topologiquement mélangeante *si pour tout ouvert* $U \subset M$ *il existe* $n \geq 0$ *tel que* $T^n(U) = M$.

Pour une application T continue uniformément dilatante et mélangeante, on a le résultat de "mélange uniforme" suivant :

Proposition 2.4
Soit $T : M \to M$ *une application continue, uniformément dilatante et topologiquement mélangeante. Alors pour tout* $R > 0$, *il existe* $N = N(R)$ *tel que* $T^N(B(x,R)) = M$ *pour tout* $x \in M$.

2.1.1 Applications dilatantes C^1

Nous introduisons maintenant une généralisation des notions précédentes au cadre C^r. Soit M une variété Riemannienne compacte, connexe, de dimension finie. On considère une application $T : M \to M$ de classe C^r, $r > 1$, telle qu'il existe $\lambda > 1$

$$\forall x \in M,\ \forall v \in T_xM, \|DT(x).v\| \geq \lambda \|v\| \tag{2.1.4}$$

Il n'est pas difficile de voir que la condition (2.1.4) implique (2.1.1), et donc que les résultats précédents s'appliquent.
On a le résultat suivant, qui montre que l'hypothèse de mélange devient superflue en régularité C^r pour $r > 1$.

Theorem 2.1
Soit $r > 1$. Une application uniformément dilatante C^r est topologiquement mélangeante.

Il est facile de voir que (2.1.4) est une condition ouverte dans $C^1(M)$, et donc que l'ensemble des appliations dilatantes C^r est un ouvert de $C^r(M)$. Il est remarquable qu'une propriété a priori bien plus forte, celle de *stabilité structurelle*, soit également vérifiée:

Definition 2.2
Soit M une variété Riemannienne, et soit $T \in C^r(M)$, r>1. On dit que T est structurellement stable *si pour tout $\widetilde{T}$ suffisamment proche de T en topologie C^r, il existe un homéomorphisme $h : M \to M$ tel que $\widetilde{T} = h^{-1} \circ T \circ h$.*

Theorem 2.2
L'ensemble des applications dilatantes C^r est structurellement stable.

2.2 Rappels de théorie ergodique

Lorsqu'un système exhibe la propriété (1.0.1), il est impossible d'étudier (numériquement) les orbites individuelles : en effet, la moindre erreur sur la localisation du point initial se traduira au bout d'un nombre fini d'itérations par une erreur exponentielle sur la localisation de l'orbite; or une telle erreur initiale est inévitable, ne serait-ce qu'en raison de la précision finie des ordinateurs.

Les limitations que cette impossibilité pratique impose expliquent pourquoi il est pertinent d'étudier les dynamiques chaotiques d'un point de vue probabiliste, i.e de considérer l'évolution des mesures de probabilités sous l'effet de la dynamique au lieu de considérer des orbites individuelles (le point de vue topologique). Cette approche mène à définir **l'opérateur de transfert**, objet d'une importance fondamentale dans ce qui va suivre : il s'agit de l'opérateur linéaire $\mathcal{L}_T$, agissant sur l'espace $\mathcal{P}(M)$ des mesures de probabilités de M, et défini pour un borélien $A \subset M$ par

$$\mathcal{L}_T \mu(A) := \mu(T^{-1}(A)) \tag{2.2.1}$$

En d'autres termes, nous considérons l'évolution de "configurations de points" au lieu de l'évolution d'une orbite individuelle.
Adopter ce point de vue sur la dynamique permet d'"échanger" une dynamique potentiellement "compliquée" (car chaotique) mais agissant sur un espace à la topologie simple (une variété Riemannienne de dimension finie), contre une dynamique "simple" car linéaire, mais agissant sur un espace à la topologie "compliquée" (un espace de Banach de dimension infinie). L'usage de l'opérateur de transfert vient de la mécanique statistique; la terminologie "transfert" découle de la manière dont cet opérateur "encode" dans ces propriétés spectrales les propriétés statistiques de la dynamique sous-jacente : nous essaierons d'expliquer comment dans cette introduction. Dans

cette perspective, les objets d'intérêt sont les *mesures invariantes*, c'est-à-dire les $\mu \in \mathcal{P}(X)$ vérifiant

$$\mathcal{L}_T\mu = \mu \tag{2.2.2}$$

Lorsque nous parlons de *propriétés statistiques*, nous parlons donc des propriétés de l'opérateur $\mathcal{L}_T$, et de celles des mesures satisfaisant (2.2.2). On notera classiquement $\mathcal{M}_T$ l'ensemble des mesures invariantes par T. Il est remarquable que lorsque M est un espace métrique compact et T est continue, l'ensemble $\mathcal{M}_T$ est non-vide, mais également convexe et compact: c'est l'objet du *théorème de Krylov-Bogolubov*.

Theorem 2.3 (Théorème de Krylov-Bogolubov)
Soit (M,d) un espace métrique compact, et soit $T : M \to M$ une application continue. Alors l'ensemble $\mathcal{M}_T := \{\mu \in \mathcal{P}(M),\ \mathcal{L}_T\mu = \mu\}$ est non-vide.
Mieux encore, si $\nu \in \mathcal{P}(M)$, alors tout point d'accumulation de $(\nu_n)_{n\geq 1}$ est dans $\mathcal{M}_T$, avec

$$\nu_n := \frac{1}{n}\sum_{k=0}^{n-1}(\mathcal{L}_T)^n\nu$$

Etant donné $T : M \to M$, les mesures invariantes permettent de donner une information asymptotique sur la probabilité de répartition des orbites. C'est le fameux *théorème ergodique de Birkhoff*, analogue dynamique de la loi des grands nombres en probabilités.

Theorem 2.4 (Théorème ergodique de Birkhoff)
Soit $(M,\mathcal{F},\mu)$ un espace de probabilité et soit $T : M \to M$ un système dynamique préservant μ. Alors pour toute $\phi \in L^1(M,\mu)$, il existe un ensemble de μ-mesure pleine sur lequel

$$\frac{1}{n}\sum_{k=0}^{n-1}\phi\circ T^k(x) \underset{n\to\infty}{\longrightarrow} \mathbb{E}[\phi|\mathcal{F}_T](x) \tag{2.2.3}$$

où $\mathcal{F}_T$ désigne la σ-algèbre des sous-ensembles T-invariants, et $\mathbb{E}[\phi|\mathcal{F}_T]$ est l'espérance conditionnelle de ϕ par rapport à $\mathcal{F}_T$.
La convergence précédente à également lieu en topologie L^1.

Un autre résultat dans la même veine, mais plus général, concerne le comportement asymptotique des moyennes de cocycles sous-additifs : c'est le *théorème ergodique sous additif de Kingman*.

Theorem 2.5 (Théorème de Kingman)
Soit $(M,\mathcal{F},\mu)$ un espace de probabilité, soit $T : M \to M$ préservant μ et $(\phi_n)_{n\in\mathbb{N}}$ une suite de fonctions mesurables vérifiant la propriété de T-sous-additivité suivante :

$$\phi_{n+m} \leq \phi_n + \phi_m \circ T^n \tag{2.2.4}$$

Supposons que $\phi_1^+ = \max(0,\phi_1) \in L^1(M,\mu)$, et notons

$$\gamma := \inf_{n\in\mathbb{N}^*}\frac{1}{n}\mathbb{E}(\phi_n) \in \mathbb{R}\cup\{-\infty\}$$

alors on a :

- *La suite* $\left(\dfrac{\phi_n}{n}\right)_{n\geq 1}$ *converge* μ*-presque partout vers une fonction* $\widetilde{\phi} : M \to \mathbb{R}\cup\{-\infty\}$ *T-invariante et telle que* $\widetilde{\phi}^+ \in L^1(M,\mu)$. *De plus,*
$$\frac{1}{n}\mathbb{E}(\phi_n) \underset{n\to\infty}{\longrightarrow} \gamma = \mathbb{E}\left(\widetilde{\phi}\right)$$
- *Si* $\phi_n \in L^1(M,\mu)$ *pour tout* $n\in\mathbb{N}$, *et que* $\gamma > -\infty$, *alors* $\widetilde{\phi}\in L^1(M,\mu)$ *et la convergence précédente a également lieu en topologie* $L^1(M,\mu)$.

Les théorèmes précédents 2.3, 2.4 semblent apporter une réponse satisfaisante à la question des propriétés statistiques d'un système dynamique. Toutefois, ils ne nous donnent aucune information sur les mesures invariantes ainsi construites. En effet il peut y avoir de nombreuses mesures invariantes pour un système donné. Par exemple, si T admet une orbite périodique $(T^i(x))_{i=0\ldots p}$, alors la mesure $\mu = \frac{1}{p+1}\sum_{i=0}^{p}\delta_{T^i(x)}$ est invariante. Pourtant, en tant que combinaison linéaire de masses de Dirac, elle ne fournit aucune information sur l'espace sous-jacent !

Nous voudrions donc exiger plus de nos mesures invariantes. Parmi ces propriétés, les plus importantes sont les suivantes:

Definition 2.3

- *On dit que la mesure* $\mu\in\mathcal{M}_T$ *est* ***ergodique*** *si pour tout borélien* $A\subset M$, $T^{-1}(A) = A$ *implique* $\mu(A)\in\{0,1\}$.

- *On dit que la mesure* $\mu\in\mathcal{M}_T$ *est* ***mélangeante*** *si pour tout couple de borélien* (A,B), *on a* $\mu(T^{-n}(A)\cap B) \underset{n\to\infty}{\longrightarrow} \mu(A)\mu(B)$.

- *Soit* $\phi : \mathbb{R}\to\mathbb{R}_+$, *telle que* $\lim_{t\to\infty}\phi(t) = 0$.
On dit que les ***corrélations décroissent à vitesse*** ϕ *si pour tout couple de boréliens* (A,B), *on a*
$$\mu(T^{-n}(A)\cap B) \leq C_{A,B}\phi(n) \tag{2.2.5}$$

- *On dit que* $\mu\in\mathcal{M}_T$ *est une* ***mesure physique*** *du système si pour* $\phi\in C^0(M)$, *l'ensemble*
$$\{x\in M, \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\phi\circ T^k(x) = \int_M \phi d\mu\}$$
(appelé bassin ergodique *de* μ*) est de mesure de Lebesgue strictement positive.*

Remark 2.1

- *Pour un système ergodique tout borélien invariant par la dynamique est trivial du point de vue de la mesure : la propriété d'ergodicité est une propriété d'indécomposabilité.*
De plus, l'ergodicité permet de préciser les conclusions des théorèmes 2.4 et 2.5: si (T,μ) *est ergodique, alors* $\mathbb{E}[\phi|\mathcal{F}_T] = \int_M \phi d\mu$ *(resp.* $\widetilde{\phi} = \inf_{n\geq 1}\frac{1}{n}\mathbb{E}(\phi_n)$*) est constante dans la conclusion du theorème 2.4 (resp. théorème 2.5)*

- *Pour un système mélangeant, les évènements $T^{-n}(A)$ et B sont* asymptotiquement *indépendants : la dynamique "oublie" ces conditions initiales. Il est facile de voir que "mélange⇒ergodicité"*

- *La propriété de mélange étant lié à une convergence, il est naturel de poser la question de la vitesse de convergence: cela mène à la notion de* vitesse de mélange*, ou encore de* décroissance des corrélations*.*

- *Admettre une mesure physique est une question d'une grande importance pratique : elle signifie que les propriétés qualitatives et asymptotiques de la dynamique peuvent être observées numériquement.*

Jusqu'ici, nous avons adopté le point de vue de l'évolution des mesures sous la dynamique. Il est intéréssant de se restreindre à l'évolution des mesures à densité par rapport à une certaine mesure de référence (bien souvent la mesure de Lebesgue, mais pas toujours).

Proposition 2.5
Étant donné une application $T : M \to M$, et une mesure à densité $d\mu = \phi dm$, $\phi \in L^1(m)$, l'opérateur de transfert $\mathcal{L}_T$ agit sur ϕ par:

$$\mathcal{L}_T\phi := \frac{d\mathcal{L}_T\mu}{dm} \tag{2.2.6}$$

i.e $\mathcal{L}_T\phi$ est la dérivée de Radon-Nikodym de $\mathcal{L}_T\mu$ par rapport à m.

L'intérêt de restreindre l'action de $\mathcal{L}_T$ aux mesures à densités vient du fait qu'il agisse alors naturellement sur des espaces à la topologie *à priori* mieux comprise que celle des espaces de mesures, e.g des espaces de fonction $L^p(dm)$ (ou, comme nous le verrons plus tard, des espaces Hölder $C^r(M)$, ou encore des espaces de Sobolev $H^t_p(M)$).
L'opérateur de transfert admet d'autres expressions, parfois plus utiles:

Proposition 2.6
Soit $T : M \to M$, et soit $\mathcal{L}_T$ son opérateur de transfert. Soit $\phi \in C^0(M)$. On a l'expression suivante pour $\mathcal{L}_T\phi$:

$$\mathcal{L}_T\phi(x) = \sum_{Ty=x} \frac{\phi(y)}{|\det(DT(y))|} \tag{2.2.7}$$

La démonstration découle tout simplement de la formule de transfert en théorie de la mesure: Si $d\mu = \phi dm$, alors on a pour tout borélien $A \subset M$

$$\mathcal{L}_T\mu(A) = \mu(T^{-1}(A)) = \int_{T^{-1}(A)} \phi dm = \int_A \sum_{Ty=x} \frac{\phi(y)}{|\det(DT(y))|} dm(y) \tag{2.2.8}$$

□

L'opérateur de transfert est le dual de l'opérateur de composition par T (aussi appelé *opérateur de Koopman* dans la littérature physique) : c'est là une de ces propriétés remarquables.

Proposition 2.7
Soit $T : M \to M$ et $(\phi, \psi) \in L^1(M) \times L^\infty(M)$. On a:

$$\int_M \phi.\psi \circ T dm = \int_M \psi \mathcal{L}_T \phi dm \tag{2.2.9}$$

Une fois encore, la démonstration est élémentaire et découle de la formule de changement de variable:

$$\int_M \phi(x)\psi(T(x))dm(x) = \int_M \sum_{Ty=x'} \frac{\phi(y)}{|\det(DT(y))|}\psi(x')dm(x') \tag{2.2.10}$$

□

Remark 2.2
Nous avons jusqu'ici considéré un opérateur de transfert particulier, l'opérateur de Ruelle-Perron-Frobenius. On peut également s'intéresser à des objets plus généraux, les opérateurs de transfert à poids*, définis par*

$$\mathcal{L}_{T,g}\phi(x) := \sum_{Ty=x} g(y)\phi(y) \tag{2.2.11}$$

où $g : M \to \mathbb{R}$ est une fonction C^{r-1}. On la prendra souvent strictement positive (pour des raisons spectrales qui seront précisées plus loin). Parmi les poids les plus fréquemment utilisés, on peut citer:

- $g = \frac{1}{|\det(DT(.))|}$*, l'inverse du Jacobien de T, qui donne l'opérateur de Ruelle-Perron-Frobenius que nous avons vu plus haut.*

- $g = \frac{1}{|\det(DT(.))|^s}$*, où $s \geq 0$ est un paramètre à ajuster. Cette variante du choix précédent est pertinente pour l'étude des ensembles de Julia associés à une dynamique donnée, plus exactement la dimension de Hausdorff de cet ensemble.*

- $g = 1$*. Ce choix simple est pertinent par exemple pour l'étude de la mesure d'entropie maximale.*

Il est bien connu que le spectre de l'opérateur de Koopman a des liens importants avec les propriétés statistiques de la dynamique (cf par exemple [77]). Il n'est donc pas surprenant que son opérateur dual soit lui aussi "spectralement pertinent" dans l'étude des propriétés statistiques.

Proposition 2.8
Soit $T : M \to M$, et soit $\mu \in \mathcal{M}_T$. Notons $\mathcal{K}_T$ l'opérateur de composition par T, défini sur $L^2(M, \mu)$ par

$$\mathcal{K}_T\phi := \phi \circ T \tag{2.2.12}$$

La mesure μ est T-ergodique si et seulement si 1 est valeur propre simple de $\mathcal{K}_T$.

Une preuve possible : Commençons par remarquer qu'on a toujours $1 \in \sigma(\mathcal{K}_T)$, puisque les fonctions constantes sont nécessairement fixées par $\mathcal{K}_T$.
Si μ est T ergodique, soit $f \in L^2(M, \mu)$ fixé par $\mathcal{K}_T$, et soit $a \in \mathbb{R}$, et $A = f^{-1}(\{a\})$. Alors A est invariant par T, et donc par ergodicité $\mu(A) = 0$ ou $\mu(A) = 1$. Ainsi, f est nécessairement (μ presque partout) constante.
Inversement, si 1 est valeur propre simple de $\mathcal{K}_T$, considérons un ensemble T-invariant $A \subset M$. Si $\mathbb{1}_A$ désigne l'indicatrice de cet ensemble, alors

$$\mathcal{K}_T(\mathbb{1}_A) = \mathbb{1}_A \circ T = \mathbb{1}_{T^{-1}(A)} = \mathbb{1}_A$$

Et donc par simplicité de la valeur propre 1, $\mathbb{1}_A$ est constante, i.e $\mu(A) \in \{0, 1\}$. □

Nous venons de voir que les points fixes de l'opérateur de Koopman mesurent en un sens l'ergodicité du système (T, μ). De même, les points fixes de l'opérateur de transfert jouent un rôle fondamental : à l'instar des points fixes de $\mathcal{L}_T$, ils permettent de construire les mesures invariantes du système:

Proposition 2.9
Soit $\phi \in L^1(M, m)$. Si $\mathcal{L}_T\phi = \phi$, alors $d\mu = \phi dm$ est une mesure invariante du système. Si de plus $\phi > 0$, cette mesure est physique.

La preuve est élémentaire; si $\phi \in L^1(M, m)$ est un point fixe de $\mathcal{L}_T$, alors on a:

$$\mu(T^{-1}(A)) = \int_{T^{-1}(A)} \phi dm = \int_M \phi \mathbb{1}_A \circ T dm = \int_M \mathcal{L}_T \phi \mathbb{1}_A dm = \int_A \phi dm = \mu(A) \tag{2.2.13}$$

et donc $d\mu = \phi dm$ est bien T-invariante.
Le caractère physique vient de l'absolue continuité de la mesure μ. En effet, d'après le théorème ergodique de Birkhoff 2.4, pour tout $\psi \in L^1(M, \mu)$, on a sur un ensemble A de μ-mesure pleine

$$\frac{1}{N} \sum_{n=0}^{N-1} \psi \circ T^k(x) \underset{N\to\infty}{\longrightarrow} \int_M \psi d\mu$$

Or $m(A) > 0$ puisque $\phi > 0$. □

On termine cette section par le théorème de Krzyzewski et Szlenk [52], assurant que pour une application dilatante suffisament régulière, il existe une mesure invariante, absolument continue et mélangeante:

Theorem 2.6
Soit $T : M \to M$ une application C^r, $r > 1$, λ-dilatante sur une variété Riemannienne compacte et connexe. T admet une unique mesure de probabilité invariante et absolument continue, $\mu = hdm$, de densité positive $h \in C^{r-1}(M)$, qui de plus est mélangeante.

Nous prouvons ce théorème dans la section 2.4, en étudiant les propriétés spectrales de l'opérateur de transfert.

Remark 2.3
- *Le mélange est en fait exponentiel : cela résulte de la propriété de trou spectral (cf.(5.2.19)).*
- *Il est indispensable de supposer une régularité $r > 1$. En effet, il est possible de construire des contre-exemples au thèorème précédent si $r = 1$ [36].*

2.3 Hölder spaces $(C^r)_{r\geq 0}$

Most of the functional analysis (see e.g Theorem 2.9 or appendix B) we present in this document takes place in the scale of Hölder spaces $(C^r(M))_{r\geq 0}$. As such we recall a few basic facts on those spaces.

Let $U \subset \mathbb{R}^n$ an open subset. Let $f \in C^0(U)$, $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $r = k + \alpha$. We say that f is a C^r map on U if f is of class C^k on U and its k-th differential (seen as a k-multilinear map) is α-Hölder. We endowed the space of C^r maps of U with the norm

$$\|f\|_{C^r} = \max(\|f\|_{C^k}, \sup_{x \neq y} \frac{\|D^k f(x) - D^k f(y)\|}{\|x-y\|^\alpha}) \tag{2.3.1}$$

It is a well-established fact that $(C^r(U), \|.\|_{C^r})$ is a Banach space.

For U an open set in $\mathbb{R}^n$, and $0 \leq \beta < \alpha < 1$, one has the compact embedding :

$$C^{k+\alpha}(U) \Subset C^{k+\beta}(U)$$

The proof of this compact embedding relies on the Arzelà-Ascoli Theorem and the following interpolation inequality :

Theorem 2.7
Let E,F be Banach spaces, $\mathcal{U} \subset E$ an open subset. Let $0 \leq \alpha < \beta < \gamma < 1$ and $k \in \mathbb{N}$. Denote by $\mu = \frac{\gamma-\beta}{\gamma-\alpha}$. Then for every $f \in C^{k+\gamma}(\mathcal{U}, F)$, one has

$$\|f\|_{C^{k+\beta}} \leq M_\alpha \|f\|^{\mu}_{C^{k+\alpha}} \|f\|^{1-\mu}_{C^{k+\gamma}} \tag{2.3.2}$$

We refer to [15] for a proof.

It is an unfortunate fact that the space of smooth functions $C^\infty(U)$ is *not dense* in the Hölder spaces $C^r(U)$ for non-integer $r > 0$. This limits the range of tools we may use in our spectral analysis of transfer operators (cf. lemma 4.2); to overcome this issue we introduce the *little Hölder space*

$$c^r(U) := \overline{C^\infty(U)}^{\|.\|_{C^r(U)}} \tag{2.3.3}$$

By construction, smooth functions are dense in the little Hölder space. The drawback here is that $c^r(U) \subsetneq C^r(U)$, so that interesting functions may be missing from the little Hölder spaces.

2.4 Spectra of expanding maps on Hölder spaces

In this section, we will study the weighted transfer operator defined by

$$\mathcal{L}\phi(x) = \sum_{y, Ty=x} g(y)\phi(y) \tag{2.2.11}$$

It is remarkable that one can link statistical properties of the dynamic to spectral properties of $\mathcal{L}$ acting on an appropriate Banach space ([4, 57, 6]). As a result, the spectral picture of transfer operators for expanding maps has been thoroughly investigated, in the works of David Ruelle [63, 64], Carlangelo Liverani[55, 57], the 2000 monograph by Viviane Baladi [4], or in a 2003 paper by Gundlash and Latushkin [41].

For example, we saw that fixed points of the transfer operator are physical measures, as soon as those fixed points live in functional spaces with good properties. Linear response formulas can also be computed from spectral data of the transfer operator ([65, 67, 57, 46, 6])

Another motivating fact in the study of transfer operator's spectral properties is that decay of correlations can be linked to convergence of $\mathcal{L}^n$ towards its spectral projectors ([55, 4]). Given $d\mu = hdm$ a fixed point of $\mathcal{L}_T$ on $L^2(M, m)$, we define the *correlation function* for $\phi, \psi \in L^2(M, \mu)$ by

$$C_{\phi,\psi}(n) := \int_M \phi \circ T^n \psi d\mu - \int_M \phi d\mu \int_M \psi d\mu \tag{2.4.1}$$

It follows from the definition and the duality property of the transfer operator (2.2.9) that

$$C_{\phi,\psi}(n) = \int_M \phi \mathcal{L}_T^n(\psi.h)dm - \int_M \phi \left(\int_M \psi.hdm\right) hdm = \int_M \phi \left(\mathcal{L}_T^n(\psi.h) - h\int_M \psi.hdm\right) dm \tag{2.4.2}$$

So that decay of correlations can be expressed as L^2 convergence of $\mathcal{L}_T^n$ towards some rank one projector : this convergence is itself connected to the concept of *spectral gap*. The operator $\mathcal{L}$ acting on the Banach space $\mathcal{B}$ has a spectral gap if :

- There exists a simple, isolated eigenvalue λ of maximal modulus, i.e $|\lambda| = \rho(\mathcal{L}_{|\mathcal{B}})$, called the *dominating eigenvalue*.
- The rest of the spectrum is contained in a disk centered at 0 and of radius strictly smaller than $\rho(\mathcal{L}_{|\mathcal{B}})$.

In this case, one has the following decomposition :

$$\mathcal{L}\phi = \lambda\Pi(\phi) + R(\phi) \tag{2.4.3}$$

where the operators Π, R have the following properties:

- $\Pi R = R\Pi = 0$
- Π is the projector on the generalized eigenspace associated with λ.
- There exist $0 < \sigma < 1$, $C > 0$ such that $\|\lambda^{-n}R^n\|_{\mathcal{B}} \leq C\sigma^n$, i.e $\lambda \notin \sigma(R)$

A weaker notion is *quasi-compactness*: for an operator $\mathcal{L}$ acting on a Banach space $\mathcal{B}$, we define the essential spectral radius of $\mathcal{L}$ on $\mathcal{B}$, $\rho_e(\mathcal{L}_{|\mathcal{B}})$ as

$$\rho_e(\mathcal{L}_{|\mathcal{B}}) := \sup_{\rho>0}\{D(0,\rho)^c \cap \sigma(\mathcal{L}_{|\mathcal{B}}) \text{ only consists of isolated eigenvalues of finite multiplicity}\}$$

We say that $\mathcal{L}$ is *quasi-compact* on $\mathcal{B}$ when $\rho_e(\mathcal{L}_{|\mathcal{B}}) < \rho(\mathcal{L}_{|\mathcal{B}})$. For such an operator, one has the following decomposition

$$\mathcal{L} = \sum_{i=1}^{N} \lambda_i(\Pi_i + \mathcal{N}_i) + \mathcal{R} \tag{2.4.4}$$

where Π_i, $\mathcal{N}_i$, $\mathcal{R}$ have the following properties :

- Π_i, $\mathcal{N}_i$ are the eigen-projection and eigen-nilpotent associated with λ_i
- $\rho(\mathcal{R}_{|\mathcal{B}}) < \min |\lambda_i|$

- $\Pi_i\Pi_j = \Pi_j\Pi_i = \delta_{ij}\Pi_i$, $\Pi_i\mathcal{N}_i = \mathcal{N}_i\Pi_i = \mathcal{N}_i$, $\Pi_i\mathcal{R} = \mathcal{R}\Pi_i = 0$, $(\lambda_i - \mathcal{L})\Pi_i = \mathcal{N}_i$

One can also compute the essential spectral radius by the *Nussbaum formula*:

$$\rho_e(\mathcal{L}_{|\mathcal{B}}) = \liminf_{n\to+\infty} \|\mathcal{L}^n - K_n\|_{\mathcal{B}} \tag{2.4.5}$$

where the $(K_n)_{n\geq 0}$ have finite rank.

Connected to this formula is the spectral Theorem of Hennion [45], giving a quick way to estimate the essential spectral radius:

Theorem 2.8
Let $(\mathcal{B}, \|.\|)$ be a Banach space, endowed with another norm $|.|$, and let $\mathcal{L}$ be an operator, bounded for $\|.\|$, such that:

1. *$\mathcal{L} : (\mathcal{B}, \|.\|) \to (\mathcal{B}, |.|)$ is compact.*
2. *There exists $(R_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ with $\liminf_{n\to+\infty}(r_n)^{1/n} = r$, such that for every $x \in \mathcal{B}$*

$$\|\mathcal{L}^n x\| \leq R_n|x| + r_n\|x\| \tag{2.4.6}$$

Then

1. *$\rho_e(\mathcal{L}_{|\mathcal{B}}) \leq r$.*
2. *If $r < \rho(\mathcal{L}_{\mathcal{B}})$, the operator $\mathcal{L}$ is quasi-compact on $\mathcal{B}$.*

Inequalities of the form (2.4.6) are often called *Lasota-Yorke* inequalities in the litterature, in reference to the seminal paper of Lasota and Yorke [53] where they first appeared.

Lastly, we introduce the notion of *topological pressure*, which as we will see plays an essential role in the spectral properties of transfer operators:

Definition 2.4
Let $r > 1$, $T : M \to M$ be a C^r λ-expanding map and $g : M \to \mathbb{R}$ be a C^{r-1} function. We define the topological pressure [1] *by*

$$P_{top}(\log(|g|)) := \lim_{n\to+\infty} \left(\sup_{x\in M} \sum_{T^n y = x} |g|^{(n)} y \right)^{1/n} = \lim_{n\to+\infty} \left(\sup_M \mathcal{L}_{T,|g|}\mathbf{1} \right)^{1/n} \tag{2.4.7}$$

where the limit exists by sub-multiplicativity and $\mathcal{L}_{T,g}$ is the weighted transfer operator defined by (2.2.11).

In so many words, the topological pressure is a weighted version of the topological entropy. As the latter, it satisfies a *variational principle*:

$$P_{top}(\log(g)) = \max_{\mu\in\mathcal{M}_T} \left(h_\mu(T) + \int_M \log(g)d\mu \right) \tag{2.4.8}$$

Although $\mathcal{L}$ does not have a spectral gap on $L^2(M, m)$ or on $C^0(M)$ ([63]), a classical Theorem of Ruelle ([63, 64]) shows that, assuming a little more regularity for the dynamic, the transfer operator admits a spectral gap on the Banach spaces $(C^r(X))_{r>0}$.

[1] The concept originate from thermodynamic formalism. For more on this notion, we refer to [68, 77]

Theorem 2.9 (Spectrum of the transfer operator on $C^r(X)$)
Let $r > 0$, X be a compact, connected Riemann manifold, of dimension d, and let $T : X \to X$ be a C^{r+1}, expanding map, with dilation constant $\lambda > 1$ and $g : X \to \mathbb{R}$ a C^r map.
Then the transfer operator defined by (2.2.11) is bounded on the space $C^r(X, \mathbb{R})$. Furthermore,

- *We have the following estimates on its spectral radii*

$$\begin{cases} \rho_{ess}(\mathcal{L}_{|C^r}) \leq \lambda^{-r} e^{P_{top}(\log|g|)} \\ \rho(\mathcal{L}_{|C^r}) \leq e^{P_{top}(\log|g|)} \end{cases}$$

- *If g is a positive function, the operator $\mathcal{L}$ acting on C^r has a spectral gap, with dominating eigenvalue $e^{P_{top}(\log g)}$.*

  The associated eigenfunction ℓ_g for $\mathcal{L}^\star$ is a distribution of order 0, i.e a Radon measure, and the eigenfunction ϕ_g of $\mathcal{L}$, normalized by $\langle \ell_g, \phi_g \rangle = 1$ is positive.

  Finally, the measure m_g, defined by $\int f dm_g = \langle \ell_g, f\phi_g \rangle$ is an invariant probability measure for T, the so-called equilibrium state [2] *of T for $\log(g)$, i.e the measure m_g realizes the maximum in (2.4.8)*

The proof relies on estimates on the (essential) spectral radius, first established by Ruelle in [63, 64]. Those estimates were refined by Gundlash and Latushkin, in the paper [41], where they give an exact formula for the essential spectral radius of the transfer operator acting on $C^r(X)$ for $r \in \mathbb{R}_+$. Our strategy for obtaining those estimates rely on *Lasota-Yorke inequalities* and Hennion result 2.8.

Proof of Theorem 2.9 For the first part of the Theorem, we follow [41, part 2, p.7]: for every $\epsilon > 0$, one can construct an open, finite cover $(U_i)_{i \in \{1..s\}}$ of M such that :

1. For every $i \in \{1, .., s\}$, $diam(U_i) < \epsilon$: choosing ϵ small enough, one can view each U_i as subset of $\mathbb{R}^d$.
2. For every $i, j \in \{1, .., s\}$ such that $\bar{U}_j \subset T(U_i)$, there exists a unique local inverse $\psi_{i,j} : U_j \to U_i$, which is λ^{-1}-Lipschitz.
3. This construction is stable by C^1 perturbation, meaning that the same $U_1, .., U_s$ can be used for close enough $T, \widetilde{T}$ in C^1-topology.

We define a $s \times s$ matrix π by:

$$\pi_{i,j} = \begin{cases} 1 \text{ if } \bar{U}_j \subset T(U_i) \\ 0 \text{ otherwise} \end{cases}$$

For $x \in M$, $i \in \{1, \ldots, s\}$, we denote by ix the T preimage of x that lies in U_i. We call the n-tuple $i^n = i_1 \ldots i_n \in \{1, \ldots, s\}^n$ admissible if $\pi_{i_1,i_2} = \ldots = \pi_{i_{n-1},i_n} = 1$, and for each admissible

[2] The concept of equilibrium state originates from thermodynamic formalism: see Ruelle's book [68, §II, p.6, Theorem 1]

n-tuple, let $U_{i^n} = \psi_{i_1,i_2} \circ .. \circ \psi_{i_{n-1},i_n}(U_{i_n})$, and $i^n x = i_1 \dots i_n x \in U_{i^n}$ be the corresponding preimage of x under T^n.

We will work with the following version of the operator $\mathcal{L}$ (that we denote, somewhat abusively, by $\mathcal{L}$ again), acting on $\bigoplus \mathcal{C}^{1+\alpha}(U_i)$ by

$$\mathcal{L}\phi(x) = \sum_{i,\pi_{i,j}=1} g(ix)\phi(ix) \tag{2.4.9}$$

for $x \in U_j$. In this context, the formula $\mathcal{L}^n\phi(y) = \sum_{x,T^n x=y} g^{(n)}(x)\phi(x)$, where $g^{(n)} = \prod_{k=0}^{n-1} g \circ T^k$ is the cocycle generated by $g : M \to \mathbb{R}$ above T, becomes, for $x \in U_{i_n}$

$$\mathcal{L}^n\phi(x) = \sum_{i_0..i_{n-1}} g^{(n)}(i_0..i_{n-1}x)\phi(i_0..i_{n-1}x) \tag{2.4.10}$$

where the sum is over the admissible n-tuple $i_0 \dots i_{n-1}$ such that $\pi_{i_{n-1},i_n} = 1$.

Let $r = k + \alpha$, $k \in \mathbb{N}$, $\alpha \in (0,1)$.

When estimating the C^r norm of $\mathcal{L}^n\phi$, there is a variety of terms, but it follows from Hennion Theorem 2.8 that to bound the essential spectral radius on $C^r(M)$, it is enough to bound the constant appearing before the term of highest regularity.

Differentiating k times (2.4.10) for $\phi \in C^r(U_{i_n})$, one can write at $(x,v) \in TM$

$$D^k[\mathcal{L}^n(\phi)](x,v) = K^n(D^k\phi)(x,v) + R_n(\phi)(x,v) \tag{2.4.11}$$

where $R_n(\phi)$ contains derivatives of ϕ up to order $k-1$, but no derivative of order k: thus its C^r norm is bounded by a term of the form

$$c_n\|\phi\|_{C^k} \tag{2.4.12}$$

where c_n is a sub-exponential sequence, i.e there exists a constant C such that for n large enough, $c_n \leq C^n$.

By the argument preceding (2.4.11) the only term of interest to us is the one carrying the α-Hölder semi norm of $D^k\phi$. This terms reads

$$\sum_{i^n} g^{(n)}(i^n x') \frac{D^k\phi(i^n x) - D^k\phi(i^n x')}{d(x,x')^\alpha}.[DT^n(i^n x)^{-1}.v, \dots, DT^n(i^n x)^{-1}.v] \tag{2.4.13}$$

so that it is bounded by

$$|D^k\phi|_\alpha \sup_{i_n} \sup_{x \neq x' \in U_{i_n}} \sum_{i^n} \|g^{(n)}(i^n x')\| \|DT^n(i^n x)^{-1}\|^k \left[\frac{d(i^n x, i^n x')}{d(x,x')}\right]^\alpha \tag{2.4.14}$$

Thus, the version of the Lasota-Yorke inequality that we established reads, following [41, Lemma 2.2, (2.11)],

$$\|\mathcal{L}^n\phi\|_{C^r} \leq s_{n,r}|D^k\phi|_{C^\alpha} + c_n\|\phi\|_{C^k} \tag{2.4.15}$$

where

$$s_{n,r} = \sum_{i^n} \sup_{x_1,x_2\in M} |g^{(n)}(i^n x_1)|.\|(DT^n(i^n x_2))^{-1}\|^k \left[\frac{d(i^n x_1, i^n x_2)}{d(x_1,x_2)}\right]^\alpha \tag{2.4.16}$$

Using Hennion Theorem 2.8, one concludes that the essential spectral radius of $\mathcal{L}$ acting on $C^r(M)$ is less or equal than $s_r := \lim_{n\to\infty} s_{n,r}^{1/n}$.

But it is easy to see that

$$s_{n,r} := \sup_{i_n} \sup_{x,y\in U_{i_n}} \sum_{i^n} |g^{(n)}(i^n x)| \|DT^n(i^n x)^{-1}\|^k \left[\frac{d(i^n x, i^n y)}{d(x,y)}\right]^\alpha \leq \frac{1}{\lambda^{n(k+\alpha)}} \sup_{i_n} \sup_{x,y\in U_{i_n}} \sum_{i^n} |g^{(n)}(i^n x)|$$

so that, using lemma 2.4.7, one has

$$s_r \leq \frac{1}{\lambda^r} \lim_{n\to\infty} \exp\left(\frac{1}{n}\log\left(\sup_{i_n}\sup_{x,y\in U_{i_n}}\sum_{i^n}|g^{(n)}(i^n x)|\right)\right) \leq \frac{1}{\lambda^r}\exp(P(\log(|g|)))$$

which is the announced bound on the essential spectral radius [3].

To obtain the bound $\rho(\mathcal{L}_{|C^r}) \leq e^{P_{top}(\log|g|)}$, one should study in more details the term $R_n(\phi)$ appearing in (2.4.11), to give a finer bound on the sequence c_n. We refer to [63, Theorem 3.1] for the details of the proof.

This last two bounds imply that the weighted transfer operator $\mathcal{L}_{T,g}$ of a C^r expanding map is quasi-compact on the scale of Hölder spaces $(C^s(M))_{s\in(0,r-1]}$.

To obtain the spectral gap one can use cone contraction theory: based on abstract results of G.Birkhoff [9], this approach was first applied in [25] and successfully extended to the study of mixing rates by C.Liverani [55]. A complete account of those works can be found in the monographs by M.Viana [76] or by V.Baladi[4]. Let us also mention the approach of Fan and Jiang [23].
In our case, we refer to lemma 5.3, where it is shown that the transfer operator $\mathcal{L}_{T,g}$ for an expanding map contracts a regular Birkhoff cone $\mathcal{C}_{L,\vec{a}} \subset C^r_+(M)$: thus, by Krein-Rutman Theorem, [51, Theorem 6.3], the transfer operator admits a spectral gap, with maximal eigenvalue $\lambda_g = e^{P_{top}(\log(g))}$ and associated eigenvector $h_g \in Int(\mathcal{C}_{L,\vec{a}})$, which is thus a positive function.

Positivity of ℓ_g, the left eigenvector of $\mathcal{L}_{T,g}$, follows from estimate (5.2.19). This yields that

$$m_g(\phi) = \langle \ell_g, \phi h_g\rangle$$

defines a *Radon measure*. Up to choosing an appropriate normalization for the spectral data ($\langle \ell_g, h_g\rangle = 1$), m_g is a probability measure.

The T-invariance comes from the classical computation:

$$m_g(\phi\circ T) = \langle \ell_g, \phi\circ T h_g\rangle = e^{-P_{top}(\log(g))}\langle \ell_g, \mathcal{L}_{T,g}(\phi\circ T h_g)\rangle = e^{-P_{top}(\log(g))}\langle \ell_g, \phi\mathcal{L}_{T,g}(h_g)\rangle = \langle \ell_g, \phi h_g\rangle = m_g(\phi)$$

□

To end this section, we deduce from Theorem 2.9 Krzyzewski and Szlenk Theorem 2.6

[3] In fact, the first bound, $\rho_{ess}(\mathcal{L}_{|C^r}) \leq s_r$ is better. It is even possible to show that it is an equality see [41]

Proof of Theorem 2.6 Note if one chooses the weight $g = \dfrac{1}{\det(DT)}$ in (2.2.11), one obtains the Ruelle transfer operator (2.2.7); the results of Theorem 2.9 yields that it has a spectral gap, with a simple eigenvalue at 1, and an associated positive eigenfunction h. The associated eigenform is the Lebesgue measure $\ell = \int_M dm$.

Normalizing h by $\int_M hdm = 1$, one sees that the equilibrium state $\mu = h \otimes \ell$ is an invariant absolutely continuous probability measure. Mixing (and even exponential decay of correlations !) follows from the spectral gap estimate

$$\forall \phi \in C^r(M),\ \|\mathcal{L}_T^n \phi - \left(\int_M \phi dm\right) h\|_{C^r} \leq C\Delta\eta^{n-1}\|\phi\|_{C^r} \tag{5.2.19}$$

by virtue of (2.4.2). □

2.5 Stability and response to perturbations

In this section, we truly enter the heart of our subject: how does the statistical properties of a chaotic system changes when one imposes an external perturbation ? How can one quantify this change ?
We introduce here the concepts allowing us to rigorously ask those questions.

Definition 2.5
Let $(T_\epsilon)_{0<\epsilon<\epsilon_0}$, such that $\epsilon \mapsto T_\epsilon \in C^r(M)$ is a continuous map and for every $0 < \epsilon < \epsilon_0$, there exists an invariant measure μ_ϵ. The measure $\mu \in \mathcal{M}_T$ is ***statistically stable*** *if $\mu_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \mu$ in the sense of distributions, i.e if for every $\phi \in C^0(M)$,*

$$\int_M \phi d\mu_\epsilon \to \int_M \phi d\mu \text{ when } \epsilon \to 0$$

This property is very important in practical applications: it means that the invariant measure is "robust" to perturbations, so that it corresponds to some object that can be observed in reality. One can give a more quantitative version of this concept: this is the notion of *response*:

Definition 2.6
Let $(T_\epsilon) \in C^s((-\epsilon_0, \epsilon_0), C^r(M))$, with $1 < s \leq r$, such that for every $\epsilon \in (-\epsilon_0, \epsilon_0)$, T_ϵ has an invariant measure μ_ϵ.

- *The system has the* ***fractional response property*** *if $\epsilon \mapsto \mu_\epsilon$ is α-Hölder for some $\alpha \in (0, 1]$, at $\epsilon = 0$, in the sense of distributions, i.e if for every $\phi \in C^0(M)$*

$$\int_M \phi d\mu_\epsilon - \int_M \phi d\mu_0 = o(\epsilon^\alpha) \tag{2.5.1}$$

- *The system has the* ***linear response*** *property if $\epsilon \mapsto \mu_\epsilon$ is differentiable in the sense of distributions at $\epsilon = 0$, i.e if there exists a measure $\dot{\mu}$ such that for every $\phi \in C^0(M)$*

$$\frac{1}{\epsilon}\left[\int_M \phi d\mu_\epsilon - \int_M \phi d\mu_0\right] - \int_M \phi d\dot{\mu} \underset{\epsilon\to 0}{\longrightarrow} 0 \tag{1.0.2}$$

In the following, we will try to give a glimpse of what results is known and which method can be used with regards to stability and linear response for expanding systems, uniformly or not, in a deterministic or random context.
We do not pretend to be comprehensive in any way : as such, we will not mention more general hyperbolic systems, Anosov, Axiom A or partially hyperbolic: we refer the interested reader to [65, 67, 66, 17], or to Ruelle's review [69], and references therein, where the problem of linear response for systems with hyperbolicity are treated. We will neither mention the linear response question for hyperbolic flows, and refer to the paper [11].

2.5.1 Weak spectral perturbation and stability

In this section, we present results concerning the stability of two classes of expanding systems :

- Consider $[0,1]$ endowed with the Lebesgue measure m. Let $(I_i)_{i=1\dots N}$ be disjoint intervals such that $[0,1]=\bigcup_{i=1}^N I_i$, and let $T:[0,1]\to[0,1]$ be such that each $T_{|I_i}$ is C^2 and λ expanding, and $[\frac{1}{T'}]_{|I_i}$ is of bounded variation. Such a map is also called a Lasota-Yorke transformation.
- Let M be a compact, finite-dimensional, connected Riemann manifold (e.g, $M=\mathbb{S}^1$), endowed with its Lebesgue measure m. Let $r>1$ and take $T:M\to M$ a λ expanding map.

Let us start by a result in the second setting, established in [7]. It concerns (stochastic) perturbations of uniformly expanding systems in the sense of (2.1.4). The idea is to look at what happens when one chooses at each step a random map in a small C^r ball around some expanding map, and then take the radius of the ball tends to 0: does the invariant density of the perturbed map tends to the one of the original system ?

Formally the setting is as follows: Given a C^r, λ expanding map T, consider the ball $B_{C^r}(T,\epsilon)$ for some $\epsilon>0$ small enough for every $\widetilde{T}\in B_{C^r}(T,\epsilon)$ to be λ expanding. Now take a probability space $(\Omega_\epsilon,\mathbb{P}_\epsilon)$ and an invertible, measure preserving map τ_ϵ. One can then define a map $F_\epsilon:\Omega_\epsilon\to B_{C^r}(T,\epsilon)$, and look at the random product (denoting F_ω instead of $F(\omega)$ omitting the ϵ index)

$$F_\omega^{(n)}:=F_{\tau^{n-1}\omega}\circ\cdots\circ F_\omega \tag{2.5.2}$$

Using cone contraction theory (actually, it is exactly the family of cones $\mathcal{C}_{L,\vec{a}}$ from (5.2.1)), it is shown that the corresponding transfer operators

$$\mathcal{L}_{\omega,\epsilon}\phi(x):=\sum_{F_{\omega,\epsilon}y=x}\frac{\phi(y)}{\det(DF_{\omega,\epsilon}(y))}$$

are strict and uniform contractions of the $\mathcal{C}_{L,\vec{a}}$ (under some conditions on L and $\vec{a}$, see Theorem 5.3). It allows to derive a spectral gap with *uniform* bounds for the transfer operators $\mathcal{L}_{\omega,\epsilon}$, and to construct an invariant density $h_{\omega,\epsilon}\in\mathcal{C}_{L,\vec{a}}\subset C^{r-1}(M)$, for which *strong structural stability* can be shown [7, Theorem 1.1], i.e

$$\|h_\epsilon-h\|_{C^{r-1}}\underset{\epsilon\to 0}{\longrightarrow}0 \tag{2.5.3}$$

Next, we present Keller result [49] on stochastic stability for one-dimensional piecewise expanding systems. The main tool for this, and the backbone of **weak spectral perturbation** is the *Keller-Liverani* Theorem, which allows one to obtain continuity w.r.t a parameter ϵ of the discrete spectrum of a quasi-compact operator P_ϵ, in the case where the map $\epsilon \mapsto P_\epsilon$ is continuous only in relative topology:

Theorem 2.10 ([50])
Let $(\mathcal{B}, \|.\|)$ be a Banach space, endowed with $|.|$ a second norm, such that $Id : (\mathcal{B}, \|.\|) \hookrightarrow (\mathcal{B}, |.|)$ is compact. For every bounded (for the $\|.\|$ norm) operator Q, let

$$|||Q||| := \sup\{|Qf|, f \in \mathcal{B}, \|f\| \leq 1\}$$

Let $(P_\epsilon)_{\epsilon \geq 0}$ be a family of bounded (for the $\|.\|$ norm) operators such that :

1. *There exists C_1, M positive constants (independent of ϵ) such that for every $\epsilon \geq 0$, $n \in \mathbb{N}$,*
$$|P_\epsilon^n| \leq C_1 M^n$$
2. *There exists C_2, C_3 positive constants, and $m \in (0,1)$, $m < M$, such that for every $\epsilon \geq 0$, $n \in \mathbb{N}$,*
$$\|P_\epsilon^n f\| \leq C_2 m^n \|f\| + C_3 M^n |f|$$
3. *There exists a monotone, upper semi-continuous function $\tau : [0, +\infty) \to [0, \infty)$, such that $\tau(\epsilon) > 0$ when $\epsilon > 0$, $\tau(\epsilon) \underset{\epsilon \to 0}{\longrightarrow} 0$ and $|||P_\epsilon - P_0||| \leq \tau_\epsilon$*

Let $\delta > 0$, $m < r < M$, and $V_{\delta,r} := \{z \in \mathbb{C}, |z| \leq r,\ or\ d(z, \sigma(P_0)) \geq \delta\}$.
Then there are positive constants $\epsilon_0 = \epsilon(\delta, r)$, $a = a(r) > 0$ and $b = b(\delta, r) > 0$ such that for every $0 \leq \epsilon \leq \epsilon_0$, every $z \in V_{\delta,r}^c$,

$$\|(z - P_\epsilon)^{-1} f\| \leq a\|f\| + b|f| \tag{2.5.4}$$

For Lasota-Yorke type transformations, for the Banach space $\mathcal{B} = L^1[0,1]$ with weak norm $\|.\|_{BV}$ (the bounded variation norm) it is well known that there exists an absolutely continuous invariant probability measure $d\mu = hdm$, where $h \in BV$. The aforementioned result allows one to show that, given a C^1 path $\epsilon \mapsto T_\epsilon$ of such maps, the map $\epsilon \mapsto h_\epsilon \in L^1$ is Hölder at any order. More precisely, it is shown in [49] that

$$\|h_\epsilon - h_\eta\|_{L^1} \leq C|\epsilon - \eta| \log(|\epsilon - \eta|) \tag{2.5.5}$$

Another case where the Keller-Liverani spectral perturbation method apply is the setting of deterministic perturbation of expanding circle maps (see [3] for the original proof): given a family of expanding maps $T_\epsilon \in C^3(\mathbb{S}^1)$, with $\epsilon \mapsto T_\epsilon$ of class C^1, together with their invariant measures $\mu_\epsilon = h_\epsilon dm$, $h_\epsilon \in C^2(\mathbb{S}^1)$ (which exists by Theorem 2.9), one can establish the uniform Lasota-Yorke inequality (see the proof of Theorem 2.9) and the continuity in relative topology $(C^2(\mathbb{S}^1), C^1(\mathbb{S}^1))$ (see e.g § 5.3), and therefore establish strong (deterministic) stability

$$\|h_\epsilon - h_\eta\|_{C^1} \leq C|\epsilon - \eta|$$

i.e that the map $\epsilon \mapsto h_\epsilon \in C^1(\mathbb{S}^1)$ is Lipschitz (see [3] or Theorem 4.1).

2.5.2 Linear response for expanding maps of the circle : a pedestrian approach

Given the last result of Lipschitz continuity of the invariant density of an expanding map of the circle, it is natural to wonder about higher-order regularity, such as differentiability. We present an elementary approach for this question, following [57, 2]. It is indeed remarkable that one does not any need sophisticated theory to treat linear linear response in dimension 1.

As before, we will be working with a family T_ϵ of C^3 maps acting on the circle $\mathbb{S}^1$. We assume that the map $\epsilon \mapsto T_\epsilon \in C^3(\mathbb{S}^1)$ is of class C^1. We introduce as before the Ruelle-Perron-Frobenius operator $\mathcal{L}_\epsilon$, acting on $C^2(\mathbb{S}^1)$, defined by

$$\mathcal{L}_\epsilon \phi(x) := \sum_{T_\epsilon y = x} \frac{\phi(y)}{T'_\epsilon(y)} \tag{2.5.6}$$

The result is as follows:

Theorem 2.11 (Thm 2.2, [2])
The map $\epsilon \mapsto h_\epsilon \in C^1(\mathbb{S}^1)$ (i.e seeing $h_\epsilon \in C^2(\mathbb{S}^1)$ as a C^1 map) is a C^1 map at $\epsilon = 0$, and one has the following linear response formula *for its derivative*

$$[\partial_\epsilon h_\epsilon]_{\epsilon=0} = (\mathbb{1} - \mathcal{L}_0)^{-1} [\partial_\epsilon \mathcal{L}_\epsilon]_{\epsilon=0} h_0 \tag{2.5.7}$$

We give a sketch of proof for this result, and refer to Theorem 4.2 for a more detailed proof in a higher-dimensional setting. It is noteworthy that this "three step" method is qualitatively the same as ours (see chapter 4).

- **Step 1** Establish a uniform spectral gap for $\mathcal{L}_\epsilon$ on $C^1(\mathbb{S}^1)$, so that one can consider the spectral projector Π_ϵ, defined by $\Pi_\epsilon(\phi) = h_\epsilon \int_{\mathbb{S}^1} \phi dm$.
- **Step 2** Considering $\mathcal{L}_\epsilon$ as a bounded operator from $C^2(\mathbb{S}^1)$ to $C^1(\mathbb{S}^1)$, the map $\epsilon \mapsto \mathcal{L}_\epsilon$ is differentiable, with

$$\partial_\epsilon \mathcal{L}_\epsilon(\phi) = -\mathcal{L}_\epsilon\left(\frac{\phi' \partial_\epsilon T_\epsilon}{T'_\epsilon} + \frac{\phi \partial_\epsilon T'_\epsilon}{T'_\epsilon} - \frac{\phi \partial_\epsilon T_\epsilon T''_\epsilon}{T'^2_\epsilon}\right)$$

- **Step 3** Using the first two steps, one has

$$[\partial_\epsilon h_\epsilon]_{\epsilon=0} = (\mathbb{1} - \mathcal{L}_0)^{-1} (\mathbb{1} - \Pi_0) [\partial_\epsilon \mathcal{L}_\epsilon]_{\epsilon=0} h_0 \tag{2.5.8}$$

This last equation is valid in a neighborhood of $\epsilon = 0$, thanks to step 1-2. If one were to make assumptions on the form of the perturbation T_ϵ, an additional step would be required (e.g, in [2] it is assumed that $\partial_\epsilon T_\epsilon = X_\epsilon \circ T_\epsilon$ for some $X_\epsilon \in C^2(\mathbb{S}^1)$).

2.5.3 Weak spectral perturbation and response

To deal with higher dimensional expanding systems in a systematic way, one has to use a more intricate theory, namely *weak spectral perturbation*. Specifically, we are interested in the following generalization of Theorem 2.10, devised by Gouëzel and Liverani in their seminal paper [39] (see also [37] for a slightly different version of this result):

Theorem 2.12 (Theorem 8.1, [39])
Let $(\mathcal{B}_0, \|.\|_0) \hookleftarrow (\mathcal{B}_1, \|.\|_1) \ldots \hookleftarrow (\mathcal{B}_k, \|.\|_k)$, $k \in \mathbb{N}$ be a family of continuously embedded Banach spaces, let $I \subset \mathbb{R}$ be an interval containing 0, and let $(P_\epsilon)_{\epsilon \in I}$ be a family of operators acting on each $\mathcal{B}_i$, $i \in \{1, \ldots, k\}$. We assume that

1. *There exists $M > 0$, such that for every $\epsilon \in I$,*

$$\|P_\epsilon^n\|_0 \leq C_1 M^n \tag{2.5.9}$$

2. *There exists $\alpha < M$, such that for every $\epsilon \in I$, every $n \in \mathbb{N}$,*

$$\|P_\epsilon^n \phi\|_1 \leq C_2 \alpha^n \|\phi\|_1 + C_3 M^n \|\phi\|_0 \tag{2.5.10}$$

3. *There exists operators $Q_1 \ldots Q_{k-1}$ so that*

$$\forall j \in \{1, \ldots, k-1\},\ \forall i \in \{j, \ldots, k\}\ \|Q_i\|_{\mathcal{B}_i \to \mathcal{B}_{i-j}} \leq C_4 \tag{2.5.11}$$

Setting $\Delta_j(\epsilon) = P_\epsilon - P_0 - \sum_{i=1}^{j-1} t^i Q_i$, one has

$$\forall j \in \{1, \ldots, k-1\},\ \forall i \in \{j, \ldots, k\}, \|\Delta_j(\epsilon)\|_{\mathcal{B}_i \to \mathcal{B}_{i-j}} \leq C_5 \epsilon^j \tag{2.5.12}$$

Under those assumptions, and setting for $z \in V_{\delta,r}$

$$R_k(\epsilon) := \sum_{i=0}^{k-1} t^i \sum_{j, \ell_1 + \cdots + \ell_j = i} (z - P_0)^{-1} Q_{\ell_1} (z - P_0)^{-1} \ldots (z - P_0)^{-1} Q_{\ell_j} (z - P_0)^{-1} \tag{2.5.13}$$

one has

$$\|(z - P_\epsilon)^{-1} - R_k(\epsilon)\|_{\mathcal{B}_k \to \mathcal{B}_0} \leq C|\epsilon|^{k-1+\eta} \tag{2.5.14}$$

In so many words, this Theorem asserts that under a condition of uniformity on the discrete spectrum (assumptions (2.5.9), (2.5.10)) and with an appropriate Taylor expansion (assumptions (2.5.11), (2.5.12)), the resolvent of an operator having loss of regularity is C^k (in fact $C^{k-1+\eta}$) when viewed as an operator from $\mathcal{B}_k$ to B_0.
This result is one of the main inspiration for our main result, Theorem 3.1 which is a generalization to fixed points of (possibly non linear) maps having loss of regularity: for a detailed comparison, we refer to the introduction of [73].

We briefly explain how one can derive linear response for C^r (r>2) expanding maps in any dimension, using Theorem 2.12 and following [6, §2.5]. We will use the family of Banach spaces $(C^s(M))_{s>0}$. There are three main steps to follow:

- **Step 1** Establish the uniform Lasota-Yorke inequalities (2.5.9) and (2.5.10) for the transfer operator $(\mathcal{L}_\epsilon)_{0<\epsilon<\epsilon_0}$: this is done in the proof of Theorem 2.9.
- **Step 2** Establish the Taylor expansion (2.5.11), (2.5.12) for the map $\epsilon \in [0, \epsilon_0] \mapsto \mathcal{L}_\epsilon \in L(C^{r-1}, C^0)$: this is usually done with ad-hoc estimates on the composition operator acting on the proper scale of Banach space: in the case of Hölder spaces, we give a detailed study of such estimates in appendix B, in the spirit of [15]. For similar estimates on the scale of Sobolev spaces, we refer to [6, Lemma 2.39].

- **Step 3** Using the first two steps, one can derive the linear response formula for the invariant density

$$[\partial_\epsilon h_\epsilon]_{\epsilon=0} = (\mathbb{1} - \mathcal{L}_0)^{-1}(\mathbb{1} - \Pi_0)[\partial_\epsilon \mathcal{L}_\epsilon]_{\epsilon=0} h_0 \tag{2.5.15}$$

Theorem 2.12 has many other applications, as it allows to study regularity with respect to parameters for spectral data of transfer operators (possibly with weight). As we saw, this spectral data is intimately connected to quantities of dynamical interest, such as topological pressure (which is the logarithm of the top eigenvalue), Gibbs measure and variance in the central limit Theorem (which are derivatives with respect to a small complex parameter of the top eigenvalue), rate of mixing (which is given by the second biggest eigenvalue)...

For a study of how the Gouëzel-Liverani Theorem allows one to obtain this type of result, we refer to [40, §8] and to [10, 12].

In chapter 4, we explain in detail how one can recover similar results using a different approach, based on the implicit function Theorem 3.1.

2.5.4 Linear response in non-uniformly expanding systems

Until now, we only considered uniformly expanding maps, where we had both structural stability and the spectral gap, and uniform regularity on the whole phase space (i.e no discontinuities for the differential). One can wonder about what happens when one weaken those assumptions, whether it is admitting non-uniform expanding behavior (like in the case of intermittent maps (2.1.2)) or non-uniform regularity (like with the tent maps).

Failure of linear response: the example of the tent map

And indeed, one should be weary of weakening the regularity assumptions. The following example, that we take from [5], shows that linear response does not hold as soon as the system has one singularity !

For $\delta \in (1,2]$, let T_δ be the tent map on $[0,1]$ with slope δ, i.e such that $T_\delta(x) = \delta x$ if $0 \leq x \leq 1/2$ and $T_\delta(x) = \delta(1-x)$ for $1/2 < x \leq 1$.
This example is interesting for many reasons, one of them being that it is one of the rare cases where one may compute explicitly the transfer operator $\mathcal{L}_\delta$ acting on $L^1([0,1])$:

$$\mathcal{L}_\delta \phi(x) := \frac{\mathbb{1}_{[0,\delta/2]}}{\delta}\left[\phi\left(\frac{x}{\delta}\right) + \phi\left(1 - \frac{x}{\delta}\right)\right] \tag{2.5.16}$$

The presence of the indicator function $\mathbb{1}_{[0,\delta/2]}$ implies that the transfer operator of a family of tent maps cannot preserve the space of Hölder functions (except of course for $\delta = 2$). As such, the natural Banach space on which one studies its spectral properties is the space of function with bounded variations, $BV([0,1])$: this is the space of L^1 functions whose derivative in the sense of distributions is a Radon measure (i.e a distribution of order 0).
On $BV([0,1])$, it is a classical fact that $\mathcal{L}_\delta$ admits a spectral gap (see the seminal paper by Lasota and Yorke, or [56] for a approach using Birkhoff cones): therefore $\mathcal{L}_\delta$ has a simple and maximal eigenvalue at 1, with associated eigenfunction $h_\delta \in BV([0,1])$, normalized so that $\int_0^1 h_\delta(x)dx = 1$, which is the density of an invariant, absolutely continuous probability measure.

Thus, it seems that one has all the ingredients for linear response. However, this is not the case: denote by $c_n(\delta) = T_\delta^n(1/2)$ the critical orbit. We have

Theorem 2.13 ([5], Theorem 6.1)
There exists a C^1 map ϕ, with $\phi(0) = \phi(1)$, a sequence δ_k with $\lim_{k\to+\infty} \delta_k = 2$ such that $c_{k+2}(\delta_k)$ is a fixed point of T_δ^k and

$$\int \phi h_{\delta_k} dx - \int \phi h_2 dx \geq Ck(2 - \delta_k) \tag{2.5.17}$$

where $C > 0$ is a constant, $\int \phi h_2 dx = 1$ and h_δ is the invariant density of T_δ.

Chapter 3

Graded calculus

This chapter aims at building the theoretical framework to study the regularity of fixed points of maps having loss of regularity, in the sense of example (1.0.5). As explained in the introduction, this type of maps, of which composition operators are prime examples, naturally appear in a dynamical context as one tries to construct and study invariant measures through transfer operators method.

We start by introducing the notion of (differentiable) graded map between couples of Banach spaces (def 3.1), and derive from there some elementary properties. We end by establishing an abstract implicit function Theorem for those differentiable graded maps, Theorem 3.1, which is at the heart of this work. Although its notations are heavy, it is noteworthy that its proof is very simple. We give a natural generalization to several derivatives in Theorem 3.2.

The main applications of those concepts are given in chapter 4 and 5. However, it seems natural to give some elementary, *ad-hoc* applications in §3.3, the first one being a development of example 1.0.5. The second one is in the author opinion more original: it studies the fixed point regularity (w.r.t parameters) of a non-linear composition operator.

3.1 Graded Calculus : elementary properties

In this part, we present the definitions, motivating examples and first properties of "graded calculus". We will this section by an implicit function Theorem, together with applications illustrating the interest of this approach.

Given X, Y two Banach spaces, we will denote by $X \overset{\jmath}{\hookrightarrow} Y$ the situation where $X \subset Y$ and the inclusion map $\jmath : X \to Y$ is bounded.

Definition 3.1

- *Let $X = (X_0, X_1)$ et $Y = (Y_0, Y_1)$ be 2 couples of Banach spaces, such that $X_1 \overset{\jmath_X}{\hookrightarrow} X_0$, and $Y_1 \overset{\jmath_Y}{\hookrightarrow} Y_0$. Let $F_1 : X_1 \to Y_1$, $F_0 : X_0 \to Y_0$ be continuous maps.*
  If F_1, F_0 satisfy
  $$\jmath_Y \circ F_1 = F_0 \circ \jmath_X \tag{3.1.1}$$
  we will say that $F = (F_1, F_0)$ is a ***graded map****.*

- *We will call a graded map differentiable at $\phi_0 \in \jmath_X(X_1)$ if there exists a bounded operator $Q := Q_{F,\phi_1} : \jmath_X(X_1) \to Y_0$ satisfying*

$$F_0(\phi_0 + z_0) = F_0(\phi_0) + Q.z_0 + \|z_0\|_{X_0}\epsilon(z_1) \tag{3.1.2}$$

where $z_1 \in X_1$, $z_0 = \jmath(z_1) \in X_0$ and $\epsilon : X_1 \to Y_0$ are such that $\|\epsilon(z)\|_{Y_0} \underset{\|z\|_{X_1}\to 0}{\longrightarrow} 0$

Let us illustrate the interest of this construction by giving some examples.

- Let (X, Y) be two couples of Banach spaces as above and $F : X_0 \to Y_0$ such that $F(X_1) \subset Y_1$, and Fréchet-differentiable, i.e

$$F(\phi_0 + z_0) - F(\phi_0) = Df(\phi_0).z_0 + \|z_0\|_{X_0}\epsilon(z_0) \tag{3.1.3}$$

  where $\epsilon(z) \to 0$ whenever $\|z\|_{X_0} \to 0$. Then F induces a differentiable graded map on (X, Y).

  Indeed, if $z_1 \in X_1$, then $\|z_0\|_{X_0} \leq C\|z_1\|_{X_1}$ where $z_0 = \jmath_X(z_1)$, so that $z_1 \to 0$ in X_1 implies $z_0 \to 0$ in X_0, and thus $\epsilon(z_0) = \epsilon(z_1)$, hence F satisfies 3.1.2.

- Given $\mathcal{U} \subset \mathbb{R}^d$ an open, relatively compact set, we consider $Y_1 = X_1 = C^1(\mathcal{U},\mathcal{U})$ and $Y_0 = X_0 = C^0(\mathcal{U},\mathcal{U})$. Let $\psi \in X_1$. Then:

  The composition operator $C := (f, g) \mapsto f \circ g$ is a differentiable graded map on X.
  Indeed, consider $(h, k) \in X_1 \times X_1$ and $f, g \in X_1 \times X_1$. By Taylor formula at order one, one has

$$C(f + h, g + k) = (f + h) \circ (g + k) = f \circ (g + k) + h \circ (g + k) \tag{3.1.4}$$
$$= f \circ g + df \circ g.k + h \circ g + dh \circ g.k + o(k) \tag{3.1.5}$$
$$= C(f, g) + df \circ g.k + h \circ g + \|k\|_{X_0}\epsilon(h, k) \tag{3.1.6}$$

  where $\epsilon(h, k) \to 0$ when $h \to 0$ in C^1-norm. The error term has the wanted form, and furthermore we deduce that

$$Q_{C,(f,g)}(h, k) = df \circ g.k + h \circ g$$

- The "square" operator defined by $F := \phi \mapsto \phi \circ \phi$ is a differentiable graded map on X.
  Indeed, consider $h \in X_1$; for each $\phi \in X_1$, one has

$$F(\phi + h, \phi + h) = (\phi + h) \circ (\phi + h) = \phi \circ (\phi + h) + h \circ (\phi + h) \tag{3.1.7}$$
$$= \phi \circ \phi + d\phi \circ \phi.h + h \circ \phi + dh \circ \phi.h + o(h) \tag{3.1.8}$$
$$= F(\phi) + d\phi \circ \phi.h + h \circ \phi + \|h\|_{X_0}\epsilon(h) \tag{3.1.9}$$

  where $\epsilon(h) \to 0$ when $h \to 0$ in the C^1-topology. The error term has the wanted form, and furthermore we deduce that

$$Q_{F,\phi}h = d\phi \circ \phi.h + h \circ \phi$$

The next property studies the compatibility of differentiability for graded maps with some elementary operations (i.e addition, product with a scalar, product with a linear form)

Proposition 3.1

- *Let $\lambda \in \mathbb{C}$, F, G be two differentiable graded maps on (X,Y). Then $F+G$ (resp. $\lambda.F$) is still a differentiable graded map.*
- *Let F, G be two differentiable graded maps on $(X,\mathbb{R})$. Then the product $F.G$ is still a differentiable graded map.*
- *Let F be a differentiable graded map on (X,Y), and $\ell \in Y_0'$. Then the function $\langle \ell, F\rangle$, graded on $(X,\mathbb{R})$ is differentiable.*

Proof. The first two points are elementary, only the last one deserves to be made precise. Write, for $\phi_1 \in X_1$ and $z_1 \in X_1$ the Taylor expansion (3.1.2), and evaluate against ℓ:

$$\langle \ell, \mathcal{F}_0(\phi_0+z_0)\rangle = \langle \ell, F_0(\phi_0)\rangle + \langle \ell, Q_{\phi_1}.z_0\rangle + \|z_0\|_{X_0}\epsilon'(z_1) \tag{3.1.10}$$

where $\epsilon'(z_1) = \langle \ell, \epsilon(z_1)\rangle \underset{z_1\to 0}{\longrightarrow} 0$. □

Another question is to study the behaviour of composed maps under this notion of differentiation, i.e to give the correct version of the chain rule in that context.

Proposition 3.2
Let F, G be two differentiable graded maps on (X,Y) and (Y,Z) respectively, such that the map $Q_{G_0,\psi_1} : j_Y(Y_1) \subset Y_0 \to Z_0$ admits a bounded extension to Y_0 (that we will still denote $Q_{G_0,\psi_1} : Y_0 \to Z_0$). Then $G \circ F$ is a differentiable graded map on (X,Z), and one has the formula

$$Q_{G\circ F,\phi_0} = Q_{G,F_1(\phi_1)}.Q_{F,\phi_1} \tag{3.1.11}$$

Proof: Write (3.1.2) for G_0 at $\psi_0 = F_0(\phi_0) \in Y_0$, and $z_0 \in X_0$:

$$G_0(F_0(\phi_0+z_0)) - G_0(F_0(\phi_0)) = Q_{G_0,F_1(\phi_1)}.(F_0(\phi_0+z_0)-F_0(\phi_0)) + \|F_0(\phi_0+z_0)-F_0(\phi_0)\|_{Y_0}\epsilon(F_1(\phi_1+z_1)-F_1(\phi_1)) \tag{3.1.12}$$

Injecting (3.1.2) for F_0 at ϕ_0 in the former formula, one gets

$$G_0\circ F_0(\phi_0+z_0) - G_0\circ F_0(\phi_0) - Q_{G_0,F_1(\phi_1)}.Q_{F_0,\phi_1}.z_0 \tag{3.1.13}$$
$$= \|z_0\|_{X_0}Q_{G_0,F_1(\phi_1)}\epsilon'(z_1) + \|Q_{F_0,\phi_1}.z_0 + \|z_0\|_{X_0}\epsilon'(z_1)\|_{Y_0}\epsilon(F_1(\phi_1+z_1)-F_1(\phi_1)) \tag{3.1.14}$$

where the product $Q_{G_0,F_0(\phi_0)}.Q_{F_0,\phi_0}$ is licit by the extension assumption. By continuity of F_1 on X_1, the right-hand term satisfies

$$\|z_0\|_{X_0}\left[\|Q_{G_0,F_1(\phi_1)}\epsilon'(z_1)\|_{Z_0} + \|Q_{F_0,\phi_1}\|_{X_0,Y_0} + \|\epsilon'(z_1)\|_{Y_0} + \|\epsilon''(z_1)\|_{Y_0}\right] \underset{z_1\to 0}{\longrightarrow} 0$$

hence

$$G_0\circ F_0(\phi_0+z_0) - G_0\circ F_0(\phi_0) - Q_{G_0,F_1(\phi_1)}.Q_{F_0,\phi_1}.z_0 = \|z_0\|_{X_0}\epsilon(z_1) \tag{3.1.15}$$

where $\epsilon(z_1) \in Z_0 \underset{z_1\to 0}{\longrightarrow} 0$. □

Higher Order differentiability It is possible to generalize the notion of graded differentiability at higher orders, i.e to allow several successive differentiations:

Definition 3.2 (Scale of Banach spaces)
Let $n \geq 1$. A family of Banach spaces $X_0 \supset X_1 \supset \cdots \supset X_n$ is said to be a scale *if the injective linear maps $j_k : X_{k+1} \to X_k$ are bounded (i.e $0 \leq i \leq j \leq n \Leftrightarrow \|.\|_{X_j} \leq \|.\|_{X_i}$).*
We will denote a scale by $X_0 \overset{j_0}{\hookleftarrow} X_1 \overset{j_1}{\hookleftarrow} \dots \overset{j_{n-1}}{\hookleftarrow} X_n$, or simply by $(X_n, \dots, X_1, X_0)$.

Definition 3.3
Let $n \geq 1$, and let $X = (X_0, \dots, X_n)$ be as in Definition 3.2. Consider a family of continuous maps, $F = (F_0, \dots, F_n)$, such that

- *For each $k \in \{0, \dots, n\}$, $F_k : X_k \to X_k$ is continuous*
- *For each $k \in \{0, \dots, n-1\}$, $j_k \circ F_{k+1} = F_k \circ j_k$*
- *There is a bounded multi-linear map $Q^{(k)} : (j_{k-1} \circ \cdots \circ j_{n-1}(X_n))^k \to X_0$ such that for every $\phi_0 = j_0 \circ \cdots \circ j_{n-1}(\phi_n)$, $z_0 = j_0 \circ \cdots \circ j_{n-1}(z_n)$ one has*

$$F_0(\phi_0 + z_0) - F_0(\phi_0) = \sum_{k=1}^{n} Q^{(k)}_{\phi_k}[z_{k-1}, \dots, z_{k-1}] + \mathcal{R}_n(z_n) \tag{3.1.16}$$

where $z_\ell = j_\ell \circ \cdots \circ j_{n-1}(z_n)$ for $1 \leq \ell \leq n$ and $\mathcal{R}_n(z_n) = \prod_{i=0}^{n-1} \|z_i\|_i \epsilon(z_n)$ where $\|\epsilon(z_n)\|_{X_0} \to 0$

3.2 Graded calculus : an implicit function Theorem

This Theorem can be thought of as a complement to the implicit function Theorem. Besides the resemblance with [39, Thm 8.1] one can see an analogy with the Nash-Moser scheme [43], with the use of a (finite) scale of spaces.

Note that scales of spaces already appeared in [37, 39] and other previous works on spectral stability ([3, 50]).

Theorem 3.1
Let $\mathcal{B}$, X_0, X_1 be Banach spaces such that $X_0 \overset{j_0}{\hookleftarrow} X_1$.
Let $A_1 \subset X_1$ be non-empty, and $A_0 = j_0(A_1) \subset X_0$.
Let $u_0 \in \mathcal{B}$, and $\mathcal{U}$ be a neighborhood of u_0 in $\mathcal{B}$.
Consider maps $F_i : \mathcal{U} \times A_i \to A_i$, where $i \in \{0, 1\}$, with the following property :

$$F_0(u, j_0(\phi_1)) = j_0 \circ F_1(u, \phi_1) \tag{3.2.1}$$

for all $u \in \mathcal{U}$, $\phi_1 \in A_1$.

Moreover, we will assume that :

(i) For every $u \in \mathcal{U}$, $F_1(u, .) : A_1 \to A_1$ admits a fixed point $\phi_1(u) \in A_1$.
Furthermore, the map $u \in \mathcal{U} \longmapsto \phi_1(u) \in X_1$ is continuous.

(ii) Let $\phi_0(u) = j_0(\phi_1(u))$.
For some $(u_0, \phi_0(u_0)) = (u_0, \phi_0) \in \mathcal{U} \times j_0(A_1)$, there exists $P_0 = P_{u_0,\phi_0} \in L(\mathcal{B}, X_0)$, $Q_0 = Q_{u_0,\phi_0} \in L(j_0(X_1), X_0)$, such that

$$F_0(u_0 + h, \phi_0 + z_0) - F_0(u_0, \phi_0) = P_0.h + Q_0.z_0 + (\|h\|_{\mathcal{B}} + \|z_0\|_{X_0})\epsilon(h, z_1) \tag{3.2.2}$$

where $h \in \mathcal{B}$ satisfies $u_0 + h \in \mathcal{U}'$, $z_1 \in A_1$, $z_0 = j_0(z_1) \in A_0$, and $\epsilon(h, z_1) \xrightarrow[(h,z_1)\to(0,0)]{X_0} 0$.

(iii) $\mathbb{1} - Q_0 \in L(j_0(X_1), X_0)$ can be extended to a bounded, invertible operator of X_0 into itself.

Then the following holds :

(i)' Let $\phi_0(u) = j_0(\phi_1(u))$. The map $u \in \mathcal{U} \mapsto \phi_0(u) \in X_0$ is Fréchet differentiable at $u = u_0$ [1].

(ii)' Its differential satisfies

$$D_u\phi(u_0) = (Id - Q_0)^{-1}P_0 \tag{3.2.3}$$

Remark 3.1
If one were to take $\epsilon(h, z_1)$ in (3.2.2) depending only upon h, one could recover a condition similar to [39, §8.1, (8.3)] (see lemma 4.1)

It can seem artificial to assume continuity of the map $u \in \mathcal{U} \longmapsto \phi_1(u) \in X_1$ without further explanation. One of the cases where such an assumption is automatically satisfied is when one of the iterates of $F_1 : \mathcal{U} \times A_1 \to A_1$, say F_1^n is a *contraction* w.r.t its second variable, a classical result in fixed point theory:

Proposition 3.3
Let $\mathcal{B}$, X be Banach spaces, $\mathcal{U} \subset \mathcal{B}$ an open set and $A \subset X$ be closed, non-empty. Let $F : \mathcal{U} \times A \to A$ be a continuous map, such that there exists $n \in \mathbb{N}$ for which F^n is a contraction with respect to its second variable.
Then for every $u \in \mathcal{U}$, $F(u, .)$ admits a unique fixed point $\phi_u \in A$, and furthermore the map $u \in \mathcal{U} \mapsto \phi_u \in X$ is continuous.

Proof of proposition 3.3: We can apply the Banach contraction principle to $F^n : \mathcal{U} \times A \to A$, and thus obtain the existence of a fixed point $\phi(u) \in A$ for every $u \in \mathcal{U}$. We also have :

$$\begin{aligned}
\|\phi(u) - \phi(u_0)\|_X &= \|F^n(u, \phi(u)) - F^n(u_0, \phi(u_0))\|_X && (3.2.4)\\
&= \|F^n(u, \phi(u)) - F^n(u_0, \phi(u)) + F^n(u_0, \phi(u)) - F^n(u_0, \phi(u_0))\|_X && (3.2.5)\\
&\leq C\|\phi(u) - \phi(u_0)\|_X + \|F^n(u, \phi(u)) - F^n(u_0, \phi(u))\|_X && (3.2.6)
\end{aligned}$$

with $C < 1$, so that :

$$\|\phi(u) - \phi(u_0)\|_X \leq \frac{1}{1 - C}\|F^n(u, \phi(u)) - F^n(u_0, \phi(u))\|_X \tag{3.2.7}$$

[1] i.e there exists a bounded, linear operator $D_u\phi_0(u_0) : \mathcal{B} \to X_0$ such that

$$\|\phi_0(u_0 + h) - \phi_0(u_0) - D_u\phi_0(u_0).h\|_{X_0} = o(\|h\|_{\mathcal{B}})$$

for all $h \in \mathcal{B}$ such that $u_0 + h \in \mathcal{U}$.

We can now conclude with the continuity of $F : \mathcal{U} \times A \to A$.

Remark that if we were to demand a stronger condition on the regularity of F with respect to u, say Hölder-continuity or Lipschitz continuity, the fixed point map $u \in \mathcal{U} \to \phi(u) \in X$ would mirror that condition. $\square$

3.2.1 Taking the first derivative : a proof of Theorem 3.1

Thanks to assumption (ii), we can estimate the difference $z_0(h) = \phi_0(u_0+h) - \phi_0(u_0)$ for $h \in \mathcal{B}$, $u_0 + h \in \mathcal{U}$.

$$\begin{aligned}\phi_0(u_0+h) - \phi_0(u_0) &= F_0(u_0+h, \phi_0(u_0+h)) - F_0(u_0, \phi_0)\\ &= F_0(u_0+h, \phi_0(u_0) + z_0(h) - F_0(u_0, \phi_0)\\ &= P_0.h + Q_0.z_0(h) + (\|h\|_{\mathcal{B}} + \|z_0(h)\|_{X_0})\epsilon(h, z_1)\end{aligned}$$

thus, by (iii):

$$z_0(h) = (Id - Q_0)^{-1}P_0.h + (Id - Q_0)^{-1}(\|h\|_{\mathcal{B}} + \|z_0(h)\|_{X_0})\epsilon(h, z_1) \tag{3.2.8}$$

Now, remark that :

- By continuity of $u \in \mathcal{U} \to \phi_1(u) \in X_1$ (which is assumption (i)), we have $\lim_{h\to 0} z_1(h) = 0$ in X_1, so that $\epsilon(h, z_1(h)) = \epsilon(h) \to 0$ in X_0 as $h \to 0$ in $\mathcal{B}$.
- $(Id - Q_0)^{-1}\epsilon(h, z_1)\|h\|_{\mathcal{B}} = o(h)$ in X_0 as $h \to 0$ in $\mathcal{B}$
- For h small enough in $\mathcal{B}$-norm,

$$\|(Id - Q_0)^{-1}\|.\|\epsilon(h)\|_{X_0} \leq \frac{1}{2} \tag{3.2.9}$$

Thus, taking the X_0-norm in (3.2.8) and choosing h small enough in $\mathcal{B}$-norm, we obtain :

$$\begin{aligned}\|z_0(h)\|_{X_0} &\leq \|(Id - Q_0)^{-1}P_0.h\|_{X_0} + \|(Id - Q_0)^{-1}\epsilon(h, z_1)\|_{X_0}\|h\|_{\mathcal{B}} + \frac{1}{2}\|z_0(h)\|_{X_0}\\ \frac{1}{2}\|z_0(h)\|_{X_0} &\leq \|(Id - Q_0)^{-1}P_0.h\|_{X_0} + \|(Id - Q_0)^{-1}\epsilon(h, z_1)\|_{X_0}\|h\|_{\mathcal{B}}\end{aligned} \tag{3.2.10}$$

and thus :

$$z_0(h) = \mathcal{O}(h) \tag{3.2.11}$$

Following (3.2.11), the second term of the sum in the right hand term of (3.2.8) becomes :

$$(Id - Q_0)^{-1}(\|h\|_{\mathcal{B}} + \mathcal{O}(h))\epsilon(h) = o(h) \tag{3.2.12}$$

Finally, in the X_0-topology,

$$z_0(h) = (Id - Q_0)^{-1}P_0.h + o(h) \tag{3.2.13}$$

and thus $u \in \mathcal{U} \to \phi_0(u) \in X_0$ is differentiable at $u = u_0$ and

$$D_u\phi_0(u_0) = (Id - Q_0)^{-1}P_0 \tag{3.2.14}$$

□

3.2.2 Generalized implicit function Theorem

We aim to iterate this approach to differentiate further the fixed point map with respect to the parameter. In order to do so, we define a notion of an *n-graded family* as such :

Definition 3.4 (Graded family)
Let $n \geq 1$ be an integer, and consider a Banach space $\mathcal{B}$, a scale $X_0 \overset{j_0}{\hookleftarrow} X_1 \overset{j_1}{\hookleftarrow} \dots \overset{j_{n-1}}{\hookleftarrow} X_n$, $\mathcal{U} \subset \mathcal{B}$ an open subset, $A_n \subset X_n$ a non-empty subset.
For $0 \leq k < l < n$, we denote by $j_{k,l}$ the bounded linear map $j_k \circ j_{k+1} \circ .. \circ j_l : X_{l+1} \to X_k$, and by $\tilde{\jmath}_k = j_k \circ .. \circ j_{n-1} : X_n \to X_k$.

Define, for $i \in \{0, .., n-1\}$, $A_i = j_{i,n-1}(A_n)$, and maps $F_i : \mathcal{U} \times A_i \to A_i$, $i \in \{0, .., n\}$ such that :

(i) For every $u \in \mathcal{U}$, $\phi_i \in A_i$, $j_i(F_{i+1}(u, \phi_{i+1})) = F_i(u, j_i(\phi_{i+1}))$

(ii) There exists $(u, \phi_n) \in \mathcal{U} \times X_n$, such that for every $h \in \mathcal{B}$ such that $u + h \in \mathcal{U}$, every $z_n \in X_n$, such that $\phi_n + z_n \in A_n$, for every $1 \leq k \leq n$, F_{n-k} satisfies

$$F_{n-k}(u+h, \phi_{n-k}+z_{n-k}) - F_{n-k}(u, \phi_{n-k}) = \sum_{\ell=n-k+1}^{n} \sum_{\substack{(i,j)\\ i+j=\ell-(n-k)}} Q^{(i,j)}(u, \phi_\ell)[h, z_{\ell-1}] + \mathcal{R}_n(h, z_n) \tag{3.2.15}$$

where for every pair (i, j) so that $i + j = \ell$, $1 \leq \ell \leq k$, one has:

- *$Q^{(i,j)}(u, \phi_{\ell+n-k}) \in L(\mathcal{B}^i \times X^j_{\ell+n-k-1}, X_{n-k})$ is a ℓ-linear map*
- *$\mathcal{R}_n \in C^0(\mathcal{B} \times X_n, X_{n-k})$ is such that $\|\mathcal{R}_n(h, z_n)\|_{X_{n-k}} = \left(\|h\|^k_{\mathcal{B}} + \|z_{n-1}\|^k_{X_{n-1}}\right) \epsilon(h, z_n)$.*

*We call a family of maps $(F_i)_{i\in\{0,..,n\}}$ acting on $\mathcal{B}$, $X_0 \overset{j_0}{\hookleftarrow} X_1 \overset{j_1}{\hookleftarrow} \dots \overset{j_{n-1}}{\hookleftarrow} X_n$ and satisfying (i)-(ii), an **n-graded family**.*

Lemma 3.1
Let $(F_i)_{i\in\{0,..,n\}}$ be an n-graded family at (u, ϕ_n), and $1 \leq k \leq n$. Let $\phi_n \in C^0(\mathcal{U}, A_n)$ $\phi_n(u) = \phi_n$ be a map with the property that for every $\ell \in \{0, \dots, k-1\}$, the map $\phi_{n-\ell} := \tilde{\jmath}_{n-\ell}(\phi_n)$ has a Taylor expansion of order ℓ at $u \in \mathcal{U}$.
Then the map $u \in \mathcal{U} \mapsto \tilde{\jmath}_{n-k} \circ F_n(u, \phi_n(u)) \in X_{n-k}$ has the following property: for every $1 \leq m \leq k$, there exists $Q_m(u) \in L^m(\mathcal{B}, X_{n-k})$ such that for every $u \in \mathcal{U}$ where (3.2.15) holds, one has

$$\tilde{\jmath}_{n-k} \circ [F_n(u + h, \phi_n(u + h)) - F_n(u, \phi_n(u))] = \tag{3.2.16}$$
$$Q^{(0,1)}_{u,\phi_{n-k+1}(u)}.[\phi_{n-k}(u + h) - \phi_{n-k}(u)] + \sum_{m=1}^{k} Q_m(u)[h, \dots, h] + \|h\|^k_{\mathcal{B}}\epsilon(h)$$

where $\epsilon(h) \in X_{n-k} \underset{h\to 0}{\longrightarrow} 0$.

In other words, if one composes a n-graded family with a map admitting a Taylor expansion at order $k-1$, one gets a map admitting a Taylor expansion at order k, once seen in X_{n-k}. This is a generalization of the idea behind Theorem 3.1.

Proof of lemma 3.1: From (3.2.15), one can write:

$$\begin{aligned}\tilde{\jmath}_{n-k}\circ[F_n(u+h,\phi_n(u+h))-F_n(u,\phi_n(u))] &= F_{n-k}(u+h,\phi_{n-k}(u+h))-F_{n-k}(u,\phi_{n-k}(u))\\ &=\sum_{\ell=1}^{k}\sum_{\substack{(i,j)\\ i+j=k-\ell+1}}Q^{(i,j)}(u,\phi_{n-\ell+1})[h,\phi_{n-\ell}(u+h)-\phi_{n-\ell}(u)]+\mathcal{R}_n(h,\phi_n(u+h)-\phi_n(u))\end{aligned} \tag{3.2.17}$$

From our assumptions, for every $\ell\in\{1,\dots,k-1\}$, $\phi_{n-\ell}$ admits a Taylor expansion of order ℓ at $u\in\mathcal{U}$, so that one can write for every $u\in\mathcal{U}$ and $h\in\mathcal{B}$ such that $u+h\in\mathcal{U}$,

$$\phi_{n-\ell}(u+h)-\phi_{n-\ell}(u)=P_\ell^{(1)}(u).h+\cdots+P_\ell^{(\ell)}(u)[h,\dots,h]+\|h\|^\ell\epsilon(h) \tag{3.2.18}$$

with $P_\ell^{(r)}(u):\mathcal{B}^r\to X_{n-\ell}$ a r-linear bounded map, and $\epsilon(h)\in X_{n-\ell}\underset{h\to 0}{\longrightarrow}0$.
The same Taylor expansion (at order $k-1$) holds for $\phi_{n-k}=\jmath_{n-k}(\phi_{n-k+1})$ at $u\in\mathcal{U}$. Injecting (3.2.18) in (3.2.17) yields a variety of terms. Nonetheless, for $1\le m\le k$, one may specify which term are m linear in h (leaving side the term $Q^{(0,1)}_{u,\phi_{n-k+1}(u)}.[\phi_{n-k}(u+h)-\phi_{n-k}(u)]$), in the following way:

$$Q_1(u)=Q^{(1,0)}_{u,\phi_{n-k+1}} \tag{3.2.19}$$

$$Q_m(u)=\sum_{\ell=2}^{m}\sum_{\substack{(i,j)\\ i+j=\ell}}\sum_{\substack{(r_1,\dots,r_j)\\ i+r_1+\cdots+r_j=m}}Q^{(i,j)}(u,\phi_{n-(k-\ell)}(u))\left[h,P^{(r_1)}_{k-\ell+1}(u),\dots,P^{(r_j)}_{k-\ell+1}(u)\right] \tag{3.2.20}$$

It is easy to see that each of the above define a bounded m-linear map $\mathcal{B}^m\to X_{n-k}$.
Furthermore, the error term has the form $\|h\|^k\epsilon(h)$. Indeed, in (3.2.17), for fixed $1\le\ell\le k$, each of the terms of the sum of indices $i+j=k-\ell+1$ contributes to the error term by a $i+j\ell=k+(\ell-1)(j-1)\ge k$ power of $\|h\|$. By definition and continuity of ϕ_n at $u\in\mathcal{U}$, the term $\mathcal{R}_n$ also gives an error of $\|h\|^k\epsilon(h)$. This yields an error term of the announced form. □

Theorem 3.2
Let $(F_i)_{i\in\{0,..,n\}}$ be a n-graded family. Let $u_0\in\mathcal{U}$. We make the following assumptions :

- *For every $u\in\mathcal{U}$, $F_n(u,.)$ admits a fixed point $\phi_n(u)$. Furthermore, we assume that the map $u\in\mathcal{U}\longmapsto\phi_n(u)\in X_n$ is continuous.*
- *For every $0\le k\le n-1$, $\mathbb{1}-Q^{(0,1)}_{u,\phi_{k+1}(u)}$ is an invertible, bounded operator of X_k.*

Then for every $1\le k\le n$ the fixed point map $u\in\mathcal{U}\longmapsto\phi_{n-k}(u)=\tilde{\jmath}_{n-k}(\phi_n(u))\in X_{n-k}$ admits a Taylor expansion of order k at $u\in\mathcal{U}$:

$$\phi_{n-k}(u+h)-\phi_{n-k}(u)=\sum_{m=1}^{k}Q_{m,n-k}(u)[h,\dots,h]+\|h\|^k\epsilon(h) \tag{3.2.21}$$

where the bounded, m-linear maps $Q_m(u) : \mathcal{B}^m \to X_{n-k}$ are obtained recursively by (3.2.26). Furthermore, if the former Taylor expansion holds at any point in some neighborhood $\mathcal{U}'$ of u and if $u \in \mathcal{U}' \mapsto (Q^{(0,1)}(u, \phi_{n-k+1}(u)), Q_{k,n-k}(u))$ is continuous, then the fixed point map ϕ_{n-k} is k-times differentiable at u[2].

Proof of Theorem 3.2: We present a proof by finite and descending induction.

- For $k = 1$, Theorem 3.2 is simply Theorem 3.1, with

$$Q_{1,n-1}(u) = D_u \phi_{n-1}(u) = [\mathbb{1} - Q^{(0,1)}_{u,\phi_n(u)}]^{-1} Q^{(1,0)}_{u,\phi_n(u)}$$

.

- For $k = 2$, we write the Taylor expansion of F_{n-2} at $(u, \phi_{n-2}(u))$:

$$\begin{aligned}
\phi_{n-2}(u+h) - \phi_{n-2}(u) &= F_{n-2}(u+h, \phi_{n-2}(u+h)) - F_{n-2}(u, \phi_{n-2}(u)) \\
&= Q^{(1,0)}_{u,\phi_{n-1}(u)}.h + Q^{(0,1)}_{u,\phi_{n-1}(u)}.[\phi_{n-2}(u+h) - \phi_{n-2}(u)] \\
&+ \sum_{i+j=2} Q^{(i,j)}_{u,\phi_n(u)}.[h, \phi_{n-1}(u+h) - \phi_{n-1}(u)] \\
&+ \left(\|h\|^2 + \|\phi_{n-1}(u+h) - \phi_{n-1}(u)\|^2_{n-1}\right) \epsilon(h, \phi_n(u+h) - \phi_n(u))
\end{aligned} \tag{3.2.22}$$

By injecting the Taylor expansion at order 1 of ϕ_{n-1} at $u \in \mathcal{U}$, one obtains

$$\left[\mathbb{1} - Q^{(0,1)}_{u,\phi_{n-1}(u)}\right] \phi_{n-2}(u+h) - \phi_{n-2}(u) = Q^{(1,0)}_{u,\phi_{n-1}(u)}.h + \sum_{i+j=2} Q^{(i,j)}_{u,\phi_n(u)}.[h, Q_{1,n-1}(u).h] + \|h\|^2 \epsilon(h) \tag{3.2.23}$$

where the error terms $\|h\|^2\epsilon(h)$ stems from continuity of ϕ_n (and ϕ_{n-1}) at u, and the composition of the error term from the order one Taylor expansion of ϕ_{n-1} with the bounded bilinear maps $Q^{(i,j)}$, for $i + j = 2$. Together with invertibility of $\mathbb{1} - Q^{(0,1)}_{u,\phi_{n-1}(u)}$, we get the promised Taylor expansion at order two, with

$$Q_{2,n-2}(u) = \left[\mathbb{1} - Q^{(0,1)}_{u,\phi_{n-1}(u)}\right]^{-1} \cdot \sum_{i+j=2} Q^{(i,j)}_{u,\phi_n(u)}.[h, Q_{1,n-1}(u).h] \tag{3.2.24}$$

- Assume that ϕ_{n-k+1} admits a Taylor expansion of order $k - 1$ at u. By lemma 3.1, and invertibility of $\mathbb{1} - Q^{(0,1)}_{u,\phi_{n-k+1}(u)}$, one may write for ϕ_{n-k}:

$$\phi_{n-k}(u+h) - \phi_{n-k}(u) = \sum_{m=1}^{k} \left[\mathbb{1} - Q^{(0,1)}_{u,\phi_{n-k+1}(u)}\right]^{-1} Q_{m,n-k}(u).[h, \dots, h] + \|h\|^k \epsilon(h) \tag{3.2.25}$$

[2]Unfortunately, the fact that the fixed point ϕ_{n-k} admits a Taylor expansion of order k is not sufficient to guarantee the k-differentiability at u, which is why this last statement was added.

with

$$Q_{1,n-k} = Q^{(1,0)}_{u,\phi_{n-k+1}} \tag{3.2.26}$$

$$Q_{m,n-k}(u) = \sum_{\ell=2}^{m} \sum_{\substack{(i,j) \\ i+j=\ell}} \sum_{\substack{(r_1,\dots,r_j) \\ i+r_1+\cdots+r_j=m}} Q^{(i,j)}(u, \phi_{n-(k-\ell)}(u)) \left[., Q_{r_1,n-(k-\ell+1)}(u), \dots, Q_{r_j,n-(k-\ell+1)}(u) \right]$$

The last statement follows from [14, Theorem 4.1] This concludes the proof. □

3.3 First applications

In this section we present 2 elementary applications of our method to fixed points problems having loss of regularity. If those examples do not have direct dynamical meaning (as with the transfer operator example of Theorem 1.2), they provide a simple, yet non-trivial illustration of the kind of functional techniques we will use in our main applications (chapter 4 and 5).

3.3.1 An elementary example

Let $I = [-1, 1]$, and consider the Banach space $C^0(I)$. Let $0 < \epsilon < 1$, and define the family of maps $(F_u)_{u \in [-\epsilon,\epsilon]}$ by :

$$F(u, \phi)(t) = \frac{1}{2}\phi\left(\frac{t+u}{2}\right) + g(t, u) \tag{3.3.1}$$

with $g : I \times [-\epsilon, \epsilon] \to \mathbb{R}$ a non-zero C^2 map, such that $g_u = g(., u) \in B_{C^2}(0, \frac{1}{2})$ for every $u \in [-\epsilon, \epsilon]$.

Under this condition, the map F_u preserves $B_{C^2}(0, 1)$, the closed unit ball of C^2. Furthermore, it is not hard to see that for any $u \in [-\epsilon, \epsilon]$, F_u is a contraction in C^2-norm: indeed, take $\phi, \psi \in B_{C^2}(0, 1)$, one has:

$$\begin{aligned} \|F(u, \phi)'' - F(u, \psi)''\|_\infty &\leq \frac{1}{8}\|\phi - \psi\|_{C^2} \\ \|F(u, \phi)' - F(u, \psi)'\|_\infty &\leq \frac{1}{4}\|\phi - \psi\|_{C^2} \\ \|F(u, \phi) - F(u, \psi)\|_\infty &\leq \frac{1}{2}\|\phi - \psi\|_{C^2} \end{aligned} \tag{3.3.2}$$

So that one obtains $\|F(u, \phi) - F(u, \psi)\|_{C^k} \leq 1/2\|\phi - \psi\|_{C^k}$, $k \in \{0, 1, 2\}$.

Being a contraction of $B_{C^2}(0, 1)$, F_u admits a fixed point, say $\phi_2(u) \in B_{C^2}(0, 1)$, by the Banach contraction principle.

Unfortunately, $(u, \phi) \in [-\epsilon, \epsilon] \times B_{C^2}(0, 1) \mapsto F(u, \phi) \in C^2(I)$ is not continuous: indeed, taking for ϕ a C^2 bump function around u will result in a difference $\|F(u, \phi) - F(v, \phi)\|_{C^2}$ bounded from below by a non-zero constant.

Hence it is not possible to apply the implicit function Theorem. This is where our method comes into play. We establish the following:

(i) The map $(u, \phi) \in [-\epsilon, \epsilon] \times B_{C^2}(0, 1) \mapsto F(u, \phi) \in C^1(I)$ is Lipschitz continuous: by proposition 3.3, this implies that the fixed point map $u \in [-\epsilon, \epsilon] \mapsto \phi_1(u) \in C^1(I)$ is also Lipschitz continuous.

(ii) F satisfies the perturbed Taylor expansion (3.2.2) from $C^1(I)$ to $C^0(I)$

(iii) The "partial derivative" $Q_{u,\phi}$ admits an extension to $C^0(I)$, such that $\mathbb{1} - Q_{u,\phi}$ is invertible on $C^0(I)$.

We start by proving (i): Take $(u,v) \in [-\epsilon,\epsilon]^2$, and $\phi \in B_{C^2}(0,1)$. Then one has

$$|F(u,\phi)' - F(v,\phi)'|(t) \leq \left(\frac{1}{8}\|\phi\|_{C^2} + \|\partial_t g_t\|_{C^1}\right)|u-v| \tag{3.3.3}$$

$$|F(u,\phi) - F(v,\phi)|(t) \leq \left(\frac{1}{4}\|\phi\|_{C^1} + \|g_t\|_{C^1}\right)|u-v| \tag{3.3.4}$$

which indeed gives the announced Lipschitz continuity.

To prove (ii), one writes, for $u+h \in [-\epsilon,\epsilon]$ and $\phi, z \in C^1(I)$,

$$\begin{aligned} F(u+h,\phi+z) - F(u,\phi) &= \frac{1}{2}\left(\phi\left(\frac{u+h+t}{2}\right) - \phi\left(\frac{u+t}{2}\right)\right) + \frac{1}{2}z\left(\frac{u+h+t}{2}\right) + g(u+h,t) - g(u,t) \\ &= \left(\frac{1}{4}\phi'\left(\frac{u+t}{2}\right) + \partial_u g_(u,t)\right).h + \frac{1}{2}z\left(\frac{u+t}{2}\right) + o(h) + |h|\epsilon(\|z\|_{C^1}) \\ &= P_{u,\phi}.h + Q_u.z + (|h| + \|z\|_\infty)\,\epsilon(h,\|z\|_{C^1}) \end{aligned} \tag{3.3.5}$$

which is an expansion of the form (3.2.2). Note that here $Q_{u,\phi} = Q_u : z \mapsto \frac{1}{2}z\left(\frac{u+t}{2}\right)$ is independent of ϕ, and clearly admits a bounded extension to $C^0(I)$, that we still denote by Q_u.

Furthermore, one has

$$\|Q_u.z\|_\infty \leq \frac{1}{2}\|z\|_\infty$$

which implies that $\|Q_u\|_{C^0} \leq 1/2$, so that $\mathbb{1} - Q_u$ is indeed invertible (with bounded inverse) on $C^0(I)$.
This conclude our verification of the assumptions in Theorem 3.1. $\square$

3.3.2 A non linear application

We now give an application of Theorem 3.1 to the study of a fixed point of a non linear map. Note also that the parameters lie in an infinite dimensional space.

Consider the interval $I = [-1,1]$, and let $\mathcal{C}^{1,1}(I)$ be the set of C^1 map on I with Lipschitz derivative, endowed with the norm $\|f\|_{1,1} = \max(\|f\|_{C^1}, \sup_{\substack{x,y\in I \\ x\neq y}} \left|\frac{f'(x)-f'(y)}{x-y}\right|)$, which makes it a Banach space. Define the map $F : \mathcal{C}^{1,1}(I) \times \mathcal{C}^{1,1}(I) \to \mathcal{C}^{1,1}(I)$ by[3]

$$F(u,\phi) = \frac{1}{2}\phi \circ \phi + u \tag{3.3.6}$$

We will show the following:

Theorem 3.3
Let I, $\mathcal{C}^{1,1}(I)$, and $F : \mathcal{C}^{1,1}(I) \times \mathcal{C}^{1,1}(I) \to \mathcal{C}^{1,1}(I)$ be as above. One has:

[3] Note that we could replace the coefficient $1/2$ by any $\lambda < 1$ in the definition of F.

(i) Let $\mathcal{U} = B_{\mathcal{C}^{1,1}}(0, r')$ be an open ball in $\mathcal{C}^{1,1}(I)$. There is $r, r' \in (0,1)$, such that for every $u \in \mathcal{U}$, $B_{\mathcal{C}^{1,1}}(0,r)$, $F(u,.)$ is a contraction of $B_{\mathcal{C}^{1,1}}(0,r)$ in the C^1 topology: therefore it admits a fixed point $\varphi_u \in B_{\mathcal{C}^{1,1}}(0,r)$, and furthermore the map $u \in \mathcal{U} \mapsto \varphi_u \in C^1(I)$ is continuous.

(ii) F acting on the scale $(C^1(I), C^0(I))$ satisfies a development of the form (3.2.2). Therefore the map $u \in \mathcal{U} \mapsto \varphi_u \in C^0(I)$ is differentiable.

Proof of Theorem 3.3:

(i) It is a straightforward computation: for every $u \in \mathcal{U}$, one has

$$\|F(u,\phi)\|_\infty \leq \frac{\|\phi\|_\infty}{2} + \|u\|_\infty$$
$$\|D_t F(u,\phi)\|_\infty \leq \frac{\|\phi'\|_\infty^2}{2} + \|u'\|_\infty$$
$$\|D_t F(u,\phi)\|_{Lip} \leq \frac{\|\phi'\|_\infty . \|\phi'\|_{Lip}}{2}(1 + \|\phi'\|_\infty) + \|u'\|_{Lip}$$

Therefore we should choose r, r' such that $\frac{r}{2} + r' \leq r$, $\frac{r^2}{2} + r' \leq r$ and $\frac{r^2}{2}(1+r) + r' \leq r$. This conditions, which admits obvious solutions, insure us that $F(u,.)$ preserves $B_{\mathcal{C}^{1,1}}(0,r)$. From now on, we fix r, r' so that those conditions are satisfied.

We now show that $\|F(u,\phi) - F(u,\psi)\|_{C^1} \leq k\|\phi - \psi\|_{C^1}$, when $\phi, \psi \in B_{\mathcal{C}^{1,1}}(0,r)$. It is noteworthy that here, k is independent of u. One has:

$$\|F(u,\phi) - F(u,\psi)\|_\infty \leq \frac{1}{2}(1 + \|\phi'\|_\infty)\|\phi - \psi\|_{C^1}$$
$$\|D_t F(u,\phi) - D_t F(u,\psi)\|_\infty \leq \frac{1}{2}(\|\psi'\|_\infty + |\phi'|_{Lip}\|\phi'\|_\infty + \|\phi'\|_\infty)\|\phi - \psi\|_{C^1}$$

so that one need to impose the following conditions on r: $\frac{1+r}{2} < 1$, $\frac{2r+r^2}{2} < 1$.
Not only do these conditions clearly have solutions, they are also compatible with the conditions imposed on r in (i). From now on, we assume that r, r' satisfy both sets of conditions.

Thus, for every $u \in B_{\mathcal{C}^{1,1}}(0,r')$, $F(u,.) : B_{\mathcal{C}^{1,1}}(0,r) \to B_{\mathcal{C}^{1,1}}(0,r)$ is a contraction in the C^1 topology. Hence it admits a fixed point $\varphi_u \in B_{\mathcal{C}^{1,1}}(0,r)$, and the map $u \in \mathcal{U} \mapsto \varphi_u \in C^1(I)$ is continuous (and even Lipschitz) by proposition 3.3.

- One can write, for $u, h \in \mathcal{C}^{1,1}(I)$ such that $u,\ u + h \in \mathcal{U}$ and $\phi, z \in C^1(I)$,

$$F(u+h, \phi+z) - F(u,\phi) = h + \frac{1}{2}[\phi' \circ \phi . z + z \circ \phi] + (z' \circ \phi).z + \|z\|_\infty \epsilon_0(z)$$

where $\|\epsilon_0(z)\|_\infty \longrightarrow 0$ as $\|z\|_\infty \longrightarrow 0$. From there it is clear that with:

$$P_{u,\phi}.h = h$$
$$Q_{u,\phi}.z = \frac{1}{2}[\phi' \circ \phi . z + z \circ \phi]$$
$$\epsilon(h, z_1) = (z' \circ \phi).z + \|z\|_\infty \epsilon_0(z) = (z_1' \circ \phi).z_0 + \|z_0\|_\infty \epsilon_0(z_0)$$

F satisfies a development of the form (3.2.2).

To conclude, we need to establish the invertibility (and boundedness of the inverse) of $Q_{u,\phi} = Q_\phi$ on $C^0(I)$.
It is easy to see that for every $\phi \in B_{\mathcal{C}^{1,1}}(0, r)$, $\|Q_\phi.z\|_\infty \leq \frac{1}{2}(1+r)\|z\|_\infty$, so that $\|Q_\phi\|_{C^0} < 1$ whenever $r < 1$ (which is insured by the sets of conditions in (i), (ii)). Therefore its Neumann series converges in $C^0(I)$, and $Id - Q_\phi$ has a bounded inverse in $C^0(I)$ for every $\phi \in B_{\mathcal{C}^{1,1}}(0, r)$. $\square$

Chapter 4

Top eigenvalue of transfer operators: perturbations and dynamical applications

4.1 Introduction and main results

This chapter presents the results of the article [73]: we study the *linear response* problem in the context of smooth uniformly expanding maps, with the tools of Theorem 3.1.

More precisely, we illustrate the abstract Theorem 3.1 by applying it to a positive, linear transfer operator $\mathcal{L}_u$, associated with a family $(T_u)_{u\in\mathcal{U}}$ of $C^{1+\alpha}$ expanding maps on a Riemann manifold X, acting on $C^{1+\alpha}(X)$, that admit an isolated, simple eigenvalue λ_u of maximal modulus (i.e a spectral gap). For that, we work with the nonlinear map $F : \mathcal{U} \times C^{1+\alpha}(X)$, defined for $u \in \mathcal{U}$ a neighborhood of $u_0 \in \mathcal{B}$ and $\phi \notin \ker \mathcal{L}_u^* \ell_{u_0}$, by

$$F(u,\phi) = \frac{\mathcal{L}_u\phi}{\int \mathcal{L}_u\phi d\ell_{u_0}} \tag{4.1.1}$$

where ℓ_u (resp. ϕ_u) is the left (resp. right) eigenvector of $\mathcal{L}_u$, chosen so that $\int \mathcal{L}_u\phi_u d\ell_u = \lambda_u$. For $u \in \mathcal{U}$, we chose ϕ_u so that $\langle \ell_{u_0}, \phi_u\rangle = 1$ (this will prove useful in § 4.3).

This (nonlinear) renormalization originates from cone contraction theory, and has been used e.g in [70, 71]. Satisfying assumption (iii) in Theorem 3.1 is the main reason why one is lead to introduce (4.1.1): indeed, working with the naive guess $\lambda_u^{-1}\mathcal{L}_u$ (for which ϕ_u is an obvious fixed point) cannot give a bounded and invertible second partial differential, by definition of an eigenvalue... It is also worth noting that the normalized maps F satisfy condition (i) in Theorem 3.1 thanks to proposition 3.3.

The main results of this chapter are as follows:
First we establish, through a simple and direct argument, a strong statistical stability result for the invariant density of a family of expanding maps:

Theorem 4.1
For every $0 \le \beta < \alpha$, $u \in \mathcal{U}$, one has

- *$F(u,.)$ acts continuously (and even analytically) on $C_+^{1+\alpha}(X)^* := \{f \in C^{1+\alpha}(X), f \geq 0$ and $f \neq 0\}$*

- *Consider $F(u,.) : C_+^{1+\alpha}(X)^* \longmapsto C_+^{1+\beta}(X)^*$. Then $u \in \mathcal{U} \mapsto F(u,.)$ is Hölder continuous, with exponent $\gamma := \alpha - \beta$.*

- *$F(u,.)$ admits a unique fixed point $\phi(u) \in C_+^{1+\alpha}(X)^*$, and $u \in \mathcal{U} \longmapsto \phi(u) \in C^{1+\beta}(X)$ is γ-Hölder.*

We establish this result in §4.3. It is noteworthy that the method of proof is very simple, and should certainly generalize to more general hyperbolic maps with a similar spectral picture (that is a spectral gap on an appropriated Banach space, such as Anosov or Axiom A maps).

Theorem 4.1 also establishes the first assumption of Theorem 3.1, and is therefore instrumental in proving the following:

Theorem 4.2
Let $0 \leq \beta < \alpha < 1$, $u_0 \in \mathcal{B}$, $\mathcal{U}$ a neighborhood of u_0, $(T_u)_{u \in \mathcal{U}}$ be a family of $C^{1+\alpha}$, expanding maps of a Riemann manifold X. For each $u \in \mathcal{U}$, let $\mathcal{L}_u$ be a weighted transfer operator on $C^{1+\alpha}(X)$, associated with T_u, defined by (4.2.1).
Let $\lambda_u > 0$ be its dominating eigenvalue, $\phi(u) \in C^{1+\alpha}(X)$, $\ell_u \in (C^{1+\alpha}(X))^*$ be the associated eigenvectors of $\mathcal{L}_u$ and $\mathcal{L}_u^*$ respectively. We denote by Π_u the associated spectral projector, and let $R_u = \mathcal{L}_u - \lambda_u \Pi_u$ (cf Theorem 2.9).
Then the following holds true:

- *The map $u \in \mathcal{U} \longmapsto \phi(u) \in C^\beta(X)$ is differentiable.*

- *We have the following linear response formula for the derivative with respect to u at $u = u_0$:*

$$D_u\phi(u_0) = \frac{1}{\lambda_{u_0}}(Id - \lambda_{u_0}^{-1} R_{u_0})^{-1}(Id - \Pi_{u_0})\partial_u \mathcal{L}(u_0, \phi_{u_0}) \tag{4.1.2}$$

We establish this result in §4.4, by applying Theorem 3.1 to F acting on the scale $(C^{1+\beta}(X), C^\beta(X))$ for any $0 < \beta < \alpha$. We show that F satisfies to a Taylor expansion of the form (3.2.2), with (see formulas 4.4.6, 4.4.3)

$$P_0 = \frac{1}{\lambda_{u_0}}(Id - \Pi_{u_0})\partial_u \mathcal{L}_u|_{u=u_0}\phi_0$$

$$Q_0 = \frac{1}{\lambda_{u_0}}\mathcal{L}_{u_0} - \Pi_{u_0}$$

Our strategy of proof is the following:

- We first show regularity results (Hölder and Lipschitz continuity, differentiability in the sense of (3.2.2)) for the transfer operator $\mathcal{L}_u$ acting on Hölder spaces, with respect to u: see lemma 4.1

- We then establish Theorem 4.1 by a direct argument (see § 4.3).

- We finally prove Theorem 4.2 by applying Theorem 3.1 to the map F defined by (4.1.1), acting on the scale $(C^{1+\beta}(X), C^\beta(X))$ (see § 4.4).

Let $d \geq 1$, $\epsilon > 0$, $\mathcal{U} = (-\epsilon, \epsilon)^d$, $0 < \alpha < 1$ and $(T_u)_{u \in \mathcal{U}} \in C^{1+\alpha}(X)$ be a $C^{1+\alpha}$ family of $C^{1+\alpha}$ expanding maps. For example, T_u can be a $C^{1+\alpha}$ perturbation of an original expanding map T_0: by openness of the expanding condition, T_u is also expanding for $u \in \mathcal{U}$ small enough.
Let $g : \mathcal{U} \times X \to \mathbb{R}$ be a $C^{1+\alpha}$ map. For every $u \in \mathcal{U}$, define the associated transfer operators (e.g, on $L^\infty(X)$) by

$$\mathcal{L}_u \phi(x) = \sum_{y, T_u y = x} g(u, y)\phi(y) \tag{4.2.1}$$

Recall that the spectral features of interest appear when the transfer operator acts on Hölder spaces (cf appendix 2.4). In the next proposition, we study the regularity of $\mathcal{L}_u$ with respect to the parameter u.

Lemma 4.1 (Regularity of the perturbed transfer operator)
Let $0 \leq \beta < \alpha < 1$, and $\gamma := \alpha - \beta$. Let X,$\mathcal{U}$ and $g, T_u, \mathcal{L}_u$ be as above.

- *$u \in \mathcal{U} \longmapsto \mathcal{L}_u \in L(C^{1+\alpha}(X), C^{1+\beta}(X))$ is γ-Hölder.*
  In particular, it is a continuous map.

- *For every $h \in \mathcal{B}$ such that $u_0 + h \in \mathcal{U}$, every $0 \leq \beta \leq \alpha$, we can define a bounded operator $\partial_u \mathcal{L}(u_0, .).h : C^{1+\beta}(X) \to C^\beta(X)$, such that for every $\phi \in C^{1+\beta}(X)$,*

$$\mathcal{L}(u_0 + h, \phi) - \mathcal{L}(u_0, \phi) - \partial_u \mathcal{L}(u_0, \phi).h = \|h\|_{\mathcal{B}} \epsilon(h) \tag{4.2.2}$$

  with $\epsilon(h) \underset{h \to 0}{\longrightarrow} 0$ in $C^\beta(X)$
  Furthermore, $\mathcal{L}$ satisfies (3.2.2) in Theorem 3.1, with the scale $(C^{1+\beta}(X), C^\beta(X))$.

Proof. By a standard argument (see [63, 41]), one can construct a family of open sets covering X, small enough to be identified with open sets in $\mathbb{R}^{\dim(X)}$, and such that on each of these open sets, the transfer operator is a (finite) sum of operators of the form $\mathcal{W}_u \phi := (g_u.\phi) \circ \psi_u$, with $\phi \in C^{1+\alpha}(W)$, $\psi \in C^{1+\alpha}(\mathcal{U} \times V, W)$ is a contraction in its second variable (and a local inverse branch of T_u), $g \in C^{1+\alpha}(\mathcal{U} \times W)$ with compact support, and V, W open sets in $\mathbb{R}^{\dim(X)}$.
We will apply the results of appendix B to the operators $\mathcal{W}_u$.

For the first item, one needs to estimate, for $\phi \in C^{1+\alpha}(W)$, $\|(\mathcal{W}_u - \mathcal{W}_v)\phi\|_{C^{1+\beta}} = \max(\|(\mathcal{W}_u - \mathcal{W}_v)\phi\|_{C^1}, \|D_x(\mathcal{W}_u - \mathcal{W}_v)\phi\|_{C^\beta})$.
Assume first that the weight g is independent of the parameter. Then by lemma B.1, (B.1.1),

$$\|(\mathcal{W}_u - \mathcal{W}_v)\phi\|_{C^{1+\beta}} \leq C\|\phi\|_{C^{1+\alpha}} \|u - v\|^\gamma \tag{4.2.3}$$

with $C = C(\alpha, \beta, \|g\|_{C^1}, \|\psi_u\|_{C^1}, \|\psi_u\|_{C^{1+\alpha}}, L_0, L_0', L_\alpha, L_\alpha')$.

Now if g also depends on $u \in \mathcal{U}$, computing $\|(\mathcal{W}_u - \mathcal{W}_v)\phi\|_{C^{1+\beta}}$ with $\phi \in C^{1+\alpha}$ would yield an additional term of the form $[(g(u, .) - g(v, .))\phi] \circ \psi(u, .)$, whose $C^{1+\beta}$ norm would be bounded by $C\|\phi\|_{C^{1+\alpha}}.\|u - v\|^\gamma$, with C a constant.
Thus, $u \in \mathcal{U} \mapsto \mathcal{L}_u \in L(C^{1+\alpha}(X), C^{1+\beta}(X))$ is (locally) γ-Hölder.

Let $\phi \in C^{1+\alpha}(W)$. The C^1 regularity of the inverse branches (w.r.t to u) allows one to consider the (partial) differential of W with respect to u. Again, assume for the time being that g does not depends on u. Define $\chi_u : X \to L(\mathcal{B}, TX)$ such that $D_u\psi_u = -\chi_u \circ \psi_u$, one gets :

$$\partial_u \mathcal{W}(u, \phi) = [Dg(\psi_u) \circ D_u\psi_u].\phi \circ \psi_u + g \circ \psi_u.[D\phi(\psi_u) \circ D_u\psi_u] \tag{4.2.4}$$

The previous formula defines a bounded operator $\partial_u\mathcal{W} \in L(\mathcal{B}, L(C^{1+\alpha}(W), C^\alpha(W)))$, by virtue of lemma B.2.

One can easily extend the former to $\mathcal{L}_u$, and define a "partial differential" $\partial_u\mathcal{L}$, taking value in $L(\mathcal{B}, L(C^{1+\alpha}(X), C^\alpha(X)))$. To what extend is it a "true" partial differential ? To answer that question one has to estimate $\|\mathcal{L}(u_0 + h, \phi) - \mathcal{L}(u_0, \phi) - \partial_u.\mathcal{L}(u_0, \phi).h\|_{C^\beta}$, for $\phi \in C^{1+\alpha}(X)$

Let $x \in X$. One has

$$[\mathcal{W}_{u_0+h}\phi - \mathcal{W}_{u_0}\phi - \partial_u\mathcal{W}(u_0, \phi).h](x) = (I) + (II) + (III)$$

where

$$\begin{aligned}
(I) &= \phi(\psi(u_0, x))[g(\psi(u_0 + h, x)) - g(\psi(u_0, x)) + Dg(\psi(u_0, x)) \circ \chi_{u_0}(x).h] \\
(II) &= g(\psi(u_0, x))[\phi(\psi(u_0 + h, x)) - \phi(\psi(u_0, x)) + D\phi(\psi(u_0, x)) \circ \chi_{u_0}(x).h] \\
(III) &= [\phi(\psi(u_0 + h, x)) - \phi(\psi(u_0, x))][g(\psi(u_0 + h, x)) - g(\psi(u_0, x))]
\end{aligned}$$

By lemma B.4, (B.2.1), and lemma B.2, (B.1.4) (I), (II) and (III) can be bounded as follows:

$$\begin{aligned}
\|(I)\|_{C^\beta} &\leq C\|\phi\|_{C^\beta}\|h\|^{1+\gamma}\|g\|_{C^{1+\beta}} \\
\|(II)\|_{C^\beta} &\leq C\|g\|_{C^\beta}\|h\|^{1+\gamma}\|\phi\|_{C^{1+\beta}} \\
\|(III)\|_{C^\beta} &\leq C\|h\|^2.\|\phi\|_{C^{1+\beta}}\|g\|_{C^{1+\beta}}
\end{aligned}$$

From the latter [1], it is straightforward that

$$\mathcal{L}(u_0 + h, \phi) - \mathcal{L}(u_0, \phi) - \partial_u\mathcal{L}(u_0, \phi).h = \|h\|_{\mathcal{B}}\epsilon(h, \|g\|_{C^{1+\beta}}, \|\phi\|_{C^{1+\beta}}) \tag{4.2.5}$$

where $\epsilon(h, \|g\|_{C^{1+\beta}}, \|\phi\|_{C^{1+\beta}}) = \mathcal{O}(\|h\|^\gamma_{\mathcal{B}})$.

Let us now show that $\mathcal{L}$ satisfies the Taylor expansion (3.2.2) in the assumptions of Theorem 3.1.

We start by recalling the following Taylor estimate, found in [15][2]:

Letting E,F,G be Banach spaces, $\mathcal{U} \subset E$, $V \subset F$ be open sets, $0 \leq \beta < \alpha < 1$, and $f, h \in C^{1+\beta}(\mathcal{U}, V)$ $g, k \in C^{1+\alpha}(V, G)$, one has

$$(g + k) \circ (f + h) = g \circ f + k \circ f + [dg \circ f].h + R_{g,f}(h, k) \tag{4.2.6}$$

where there exists some $0 < \rho < 1$ such that the remainder term $R_{g,f}(h, k)$ satisfies

$$\|R_{g,f}(h, k)\|_{C^\beta} \leq C(\|h\|^{1+\rho}_{C^{1+\beta}} + \|h\|_{C^{1+\beta}}\|k\|_{C^{1+\alpha}}) \tag{4.2.7}$$

[1] From the previous bounds, one can even conclude that the map $u \in \mathcal{U} \longmapsto \mathcal{L}(u, \phi) \in C^\beta(X)$ is $C^{1+\gamma}$ for $\phi \in C^{1+\alpha}(X)$, which is precisely the conclusion drawn from the Taylor expansion at first order in Gouëzel-Liverani's paper ([39, §8.1, (8.3)]).

[2] We specifically refer to estimate (6.7) after Theorem 6.10

This, together with the definition of $\partial_u \mathcal{W}_u$, yields for $(\phi, z) \in C^{1+\alpha}(W)$

$$
\begin{aligned}
&\mathcal{W}_{u_0+h}(\phi+z) - \mathcal{W}_{u_0}(\phi) - \partial_u \mathcal{W}(u_0,\phi).h - \mathcal{W}_{u_0}(z) \\
&\qquad = D(g\phi)\circ\psi_{u_0}.(\psi_{u_0+h} - \psi_{u_0} - \partial_u\psi_{u_0}.h) + R_1(\psi_{u_0+h} - \psi_{u_0}, g.z)
\end{aligned} \tag{4.2.8}
$$

where $R_1 = R_{\phi,\psi_{u_0}}$ from 4.2.7. We start by bounding the first term. One has

$$
\psi_{u_0+h} - \psi_{u_0} - \partial_u\psi_{u_0}.h = \int_0^1 [\partial_u\psi(u_0+th) - \partial_u\psi(u_0)].hdt \tag{4.2.9}
$$

which leads us to estimate a term of the form $\|df(\psi(u_0)).\int_0^1[\partial_u\psi(u_0+th) - \partial_u\psi(u_0)].hdt\|_{C^\beta}$. Following the trick used in the proof of lemma B.4, we get

$$
\begin{aligned}
&\|df(\psi(u_0)).\int_0^1 [\partial_u\psi(u_0+th) - \partial_u\psi(u_0)].hdt\|_{C^\beta} \\
&\qquad \leq [C_1\|df\|_{C^\beta}\|\psi(u_0)\|^\beta_{C^1} + C_2\|df\|_\infty]\frac{\|h\|^{1+\gamma}}{1+\gamma}
\end{aligned} \tag{4.2.10}
$$

Now for R_1 we write, following estimate (4.2.7):

$$
\|R_1\|_{C^\beta} \leq M[\|h\|^{1+\rho} + \|h\|.(C_1\|z\|_{C^{1+\alpha}} + C_0\|z\|_{C^\alpha})] \tag{4.2.11}
$$

with $C_1,\ C_2$ depending on $\alpha,\ \|g\|_{C^\alpha},\ \|g\|_{C^{1+\alpha}}$.

Therefore, we obtained the following bound for (4.2.8) :

$$
M\|h\|^{1+\rho} + M'\|h\|^{1+\gamma} + C_1'\|h\|.\|z\|_{C^{1+\alpha}} + C_2'\|h\|.\|z\|_{C^\alpha} = [\|h\| + \|z\|_{C^\alpha}]\epsilon(h, z_{1+\alpha}) \tag{4.2.12}
$$

where $z_{1+\alpha}$ is z in $C^{1+\alpha}$ topology and $\epsilon(h, z_{1+\alpha}) \underset{(h,z_{1+\alpha})\to 0}{\longrightarrow} 0$ in $C^\beta(X)$.

In the case of a weight g depending on the parameter u, the partial derivative $\partial_u\mathcal{W}$ is given by

$$
\partial_u\mathcal{W}(u,\phi) = ([D_u(g)(u)]\phi)\circ\psi(u) + D_x(g\phi)\circ\psi(u).D_u\psi(u) \tag{4.2.13}
$$

Thus, the Taylor expansion at (u_0, ϕ) now has an additional term

$$
[(g(u_0+h) - g(u_0) - D_u(g)(u_0).h)\phi]\circ\psi(u_0)
$$

This term can be bounded (in C^β-norm), with upper bound of the form $C\|g\|_{C^{1+\alpha}}\|h\|^{1+\gamma}$, where $C = C(\|\psi(u_0)\|_{C^{1+\alpha}}, \|\phi\|_{C^{1+\alpha}})$ is a constant, as outlined in lemma B.4.
It follows that the transfer operator defined in (4.2.1) also has a Taylor expansion of the form (3.2.2). □

Remark 4.1
The previous regularity results are given for $\mathcal{L}_u$ acting on the scale $(C^{1+\beta}(X),\ C^\beta(X))$, $0 < \beta < \alpha \leq 1$. Following the method outlined in [15], and using Theorem 3.2, one can show (by induction) that $\mathcal{L}_u$ acting on the scale $C^{k+\beta}(X), C^{k-j+\beta}(X)$ has a Taylor development of the form (3.2.15) at order j, with $0 \leq j < k$ integers.

4.3 Hölder continuity of the spectral data : proof of Theorem 4.1

This section is devoted to establish Theorem 4.1, by a direct argument. Note that this type of result is already known for a one-dimensional parameter, with previous works on spectral stability [3, 50], or in the context of piecewise expanding maps of the interval [49].

Let $0 \leq \beta < \alpha < 1$, and $(T_u)_{u\in\mathcal{U}}$ be a family of $C^{1+\alpha}$ expanding maps, on a Riemann manifold X. Let $g : X \to \mathbb{R}$ be a positive [3] $C^{1+\alpha}$ function.
It follows from Ruelle Theorem [63] that the transfer operator $(\mathcal{L}_u)_{u\in\mathcal{U}}$ admits a spectral gap in $C^{1+\alpha}(X)$. Let λ_u be the dominating eigenvalue of $\mathcal{L}_u$, $\phi_u \in C^{1+\alpha}(X)$ (resp $\ell_u \in (C^{1+\alpha}(X))'$) be the right (resp left) eigenvector of $\mathcal{L}_u$ associated with λ_u, chosen so that $\langle \ell_u, \phi_u\rangle = 1$. Let $F : \mathcal{U} \times C^{1+\alpha}(X)$, defined for $u \in \mathcal{U}$ and $\phi \notin \ker \mathcal{L}_u^* \ell_{u_0}$, by

$$F(u, \phi) = \frac{\mathcal{L}_u \phi}{\langle \ell_{u_0}, \mathcal{L}_u \phi\rangle} \tag{4.1.1}$$

Note that F trivially inherits every regularity property of $(u, \phi) \in \mathcal{U} \times C_+^{1+\alpha}(X)^* \longmapsto \mathcal{L}_u\phi$, so in particular it is γ-Hölder in $u \in \mathcal{U}$ when considered as an operator from $C_+^{1+\alpha}(X)^*$ to $C_+^{1+\beta}(X)^*$. Hence the first point.

The second item follows from the former remark and the fact that ℓ_{u_0} admits a bounded extension to $C^{1+\beta}(X)$, for every $0 \leq \beta < \alpha$ (cf [63]).
Let $\phi_u \in C_+^{1+\alpha}(X)^*$ be an eigenvector for λ_u, the dominating eigenvalue of $\mathcal{L}_u$. Then one has

$$F(u, \phi_u) = \frac{\lambda_u \phi_u}{\lambda_u \langle \ell_{u_0}, \phi_u\rangle} = \frac{\phi_u}{\langle \ell_{u_0}, \phi_u\rangle} \tag{4.3.1}$$

For every $u \in \mathcal{U}$, fix a $\phi_u \in \ker(\lambda_u - \mathcal{L}_u)$ such that $\langle \ell_{u_0}, \phi_u\rangle = 1$. Such a ϕ_u is unique in $\ker(\lambda_u - \mathcal{L}_u)$ and verifies

$$F(u, \phi_u) = \phi_u \tag{4.3.2}$$

so that $F(u, .)$ has a unique fixed point ϕ_u in $C_+^{1+\alpha}(X)^*$ for every $u \in \mathcal{U}$.
Remark that for every $k \in \mathbb{N}^*$, for every $u \in \mathcal{U}$, every $\phi \notin \ker((\mathcal{L}_u^*)^k \ell_{u_0})$,

$$F^k(u, \phi) = \frac{\mathcal{L}_u^k(\phi)}{\langle \ell_{u_0}, \mathcal{L}_u^k(\phi)\rangle} \tag{4.3.3}$$

by an immediate induction
Now note that, for every $k \in \mathbb{N}^*$, $u \in \mathcal{U}$,

$$\phi(u) - \phi(u_0) = F^k(u, \phi(u)) - F^k(u_0, \phi(u)) + F^k(u_0, \phi(u)) - F^k(u_0, \phi(u_0)) \tag{4.3.4}$$

and that

$$F^k(u_0, \phi(u)) - F^k(u_0, \phi(u_0)) = \frac{\mathcal{L}_{u_0}^k(\phi(u))}{\langle \ell_{u_0}, \mathcal{L}_{u_0}^k(\phi(u))\rangle} - \phi(u_0) = \lambda_{u_0}^{-k} R_{u_0}^k(\phi(u) - \phi(u_0)) \tag{4.3.5}$$

[3] Note that we only need the positivity of the weight to insure the simplicity of the maximal eigenvalue.

Recall that there is a $0 < \sigma < 1$ such that $\|\lambda_{u_0}^{-k} R_{u_0}^k\|_{C^{1+\beta}} \leq C\sigma^k$ (cf § 2.9), so that for k large enough, one has

$$\|F^k(u_0, \phi(u)) - F^k(u_0, \phi(u_0))\|_{C^{1+\beta}} \leq \frac{1}{2}\|\phi(u) - \phi(u_0)\|_{C^{1+\beta}} \tag{4.3.6}$$

From there,(4.3.4) yields

$$\|\phi(u) - \phi(u_0)\|_{C^{1+\beta}} \leq C_{k,u}\|u - u_0\|^\gamma + \frac{1}{2}\|\phi(u) - \phi(u_0)\|_{C^{1+\beta}}$$
$$\|\phi(u) - \phi(u_0)\|_{C^{1+\beta}} \leq 2C_{k,u}\|u - u_0\|^\gamma$$

where $C_{k,u} = \|F^k(., \phi(u))\|_{C^{1+\beta}}$. Thus, $u \in \mathcal{U} \longmapsto \phi(u) \in C^{1+\beta}(X)$ is γ-Hölder. $\square$

4.4 Differentiability of the spectral data : proof of Theorem 4.2

Let $0 \leq \beta < \alpha < 1$. This section is devoted to establish Theorem 4.2 by applying Theorem 3.1 to the map F from (4.1.1) acting on the scale $(C^{1+\beta}(X), C^\beta(X))$.

The first hypothesis, i.e existence, for every $u \in \mathcal{U}$, of a fixed point ϕ_u for the map $F(u, .) : C_+^{1+\alpha}(X)^* \to C_+^{1+\alpha}(X)^*$ from (4.1.1) and continuity of the map $u \in \mathcal{U} \mapsto \phi_u \in C^{1+\beta}(X)$, has already been addressed in Theorem 4.1.

We now turn to assumption (ii). We showed the perturbed Taylor development for $\mathcal{L}$ acting on $(C^{1+\beta}(X), C^\beta(X))$ in lemma 4.1 : it immediately follows that F acting on the scale $(C^{1+\beta}(X), C^\beta(X))$ satisfies the perturbed Taylor development (3.2.2).

We now check assumption (iii). We start by remarking for every $z \in C^{1+\beta}(X)$,

$$Q_{u,\phi}.z = \frac{1}{\langle \ell_{u_0}, \mathcal{L}(u, \phi)\rangle^2}[\mathcal{L}(u, z)\langle \ell_{u_0}, \mathcal{L}(u, \phi)\rangle - \mathcal{L}(u, \phi)\langle \ell_{u_0}, \mathcal{L}(u, z)\rangle] \tag{4.4.1}$$

Thus, for $\phi = \phi_u$, we obtain

$$Q_{u,\phi_u}.z = \frac{1}{\lambda_u}(\mathcal{L}(u, z) - \langle \ell_{u_0}, \mathcal{L}(u, z)\rangle\phi_u) \tag{4.4.2}$$

and for $u = u_0$:

$$Q_{u_0,\phi_{u_0}} = \frac{1}{\lambda_{u_0}}\mathcal{L}(u_0) - \Pi_{u_0} = \frac{1}{\lambda_{u_0}}R_{u_0} \tag{4.4.3}$$

where $\Pi_{u_0}z = \langle \ell_{u_0}, z\rangle\phi_{u_0}$, $z \in C^{1+\beta}(X)$ is the spectral projector on the (one-dimensional) eigenspace associated to λ_{u_0}. It is also noteworthy that the previous expression is independent of ϕ_{u_0}.
From (4.4.3), one sees that there is a $N \geq 1$ such that $\|Q_{u_0}^N\|_{C^\beta} \leq C\sigma^N$, for some $C > 0$ and $\sigma \in (0, 1)$ (cf appendix 2.4, (2.4.3)): therefore its Neumann series converges towards $(Id - Q_{u_0})^{-1}$. This proves (iii) in the assumptions of Theorem 3.1.
We can therefore conclude that

If $\phi_u \in C^{1+\beta}(X)$, $u \in \mathcal{U} \longmapsto \phi_u \in C^\beta(X)$ is differentiable.

and that its differential satisfies

$$D_u\phi(u_0) = (Id - Q_{u_0,\phi_{u_0}})^{-1}P_{u_0,\phi_{u_0}} \tag{4.4.4}$$

Furthermore,

$$P_{u,\phi} = \frac{\partial_u\mathcal{L}(u,\phi)}{\langle \ell_{u_0}, \mathcal{L}(u,\phi)\rangle} - \frac{\langle \ell_{u_0}, \partial_u\mathcal{L}(u,\phi)\rangle}{\langle \ell_{u_0}, \mathcal{L}(u,\phi)\rangle^2}\mathcal{L}(u,\phi) \tag{4.4.5}$$

which simplifies, for $(u,\phi) = (u_0, \phi_{u_0})$, to

$$\begin{aligned} P_{u_0,\phi_{u_0}} &= \frac{1}{\lambda_{u_0}}(\partial_u\mathcal{L}(u_0,\phi_{u_0}) - \langle \ell_{u_0}, \partial_u\mathcal{L}(u_0,\phi_{u_0})\rangle\phi_{u_0}) && (4.4.6)\\ &= \frac{1}{\lambda_{u_0}}(Id - \Pi_{u_0}) \circ \partial_u\mathcal{L}(u_0,\phi_{u_0}) && (4.4.7)\end{aligned}$$

This, together with (4.4.3), proves formula (4.1.2). □

4.5 Regularity for various dynamical quantities

We now derive a few consequences of Theorems 4.1 and 4.2 for various dynamical quantities of interest :

Corollary 4.1 (Regularity of the topological pressure)
We place ourselves in the same setting as in Theorem 4.2. The topological pressure $P(u)$ is differentiable with respect to the parameter, i.e the real valued map $u \in \mathcal{U} \longmapsto P(u)$ is differentiable.

Proof. Let $u_0 \in \mathcal{B}$, and $\mathcal{U} \subset \mathcal{B}$ be a neighborhood of u_0. Given the normalization chosen for ℓ_{u_0} and ϕ_u (cf chapter 4, (4.1.1)) for every $u \in \mathcal{U}$ one has

$$\lambda_u = \langle \ell_{u_0}, \mathcal{L}(u,\phi_u)\rangle \tag{4.5.1}$$

Thus, injecting (3.2.2) and using the Hölder continuity (resp differentiability) of $u \in \mathcal{U} \mapsto \phi_u \in C^{1+\beta}(X)$ (*resp* $C^\beta(X)$), one gets the differentiability of the map $u \in \mathcal{U} \mapsto \lambda_u$.
Fix a $u_0 \in \mathcal{U}$. One has from Theorem 2.9 (or from Theorem 5.3) that $\lambda_{u_0} > 0$, so that $P(u) = \log(\lambda_u)$ is well-defined and differentiable in a neighborhood of $u_0 \in \mathcal{U}$, giving the desired conclusion. □

Corollary 4.2 (Regularity of the Gibbs measure)
We place ourselves in the same setting as in Theorem 4.2. Let m_u be defined on $C^\beta(X)$ by $m_u(f) = \langle \ell_u, f\phi_u\rangle$. Then it is a Radon measure, and for every $f \in C^\beta(X)$, the map $u \in \mathcal{U} \longmapsto m_u(f)$ is C^1.
In particular, one has:

- ***Linear response for the S.R.B measure*** *$u \in \mathcal{U} \mapsto \int_M f\phi_u dm$ is C^1, with the following formula for its differential:*

$$D_u\left[\int_M f\phi_u dm\right]_{u=u_0} = \sum_{n=0}^{+\infty}\int_M f \circ T_{u_0}^n P_{u_0}\phi_{u_0} dm \tag{4.5.2}$$

- ***Regularity for the maximal entropy measure***
If $m_{top}(u)$ is the measure of maximal entropy of T_u, the map $u \in \mathcal{U} \mapsto m_{top}(u)[f]$ is C^1.

Proof. We present a standard positivity argument by which we extend continuously ℓ_u to $C^0(X)$.
Fix a $u \in \mathcal{U}$. From Theorem 5.3, the transfer operator $\mathcal{L}_u$ contracts a family of cones, $(K_a)_{a\geq 0} \subset C^\beta(X) \cup C^0_+(X)$. Consequently, the decomposition (2.4.3) holds true in $C^\beta(X)$, for every $0 < \beta \leq 1$, and one can write

$$\lambda_u^{-n}\mathcal{L}_u^n\phi \underset{n\to+\infty}{\xrightarrow{C^\beta}} \phi_u\langle \ell_u, \phi\rangle \tag{4.5.3}$$

Now, let $\phi \in C^\beta(X) \cup C^0_+(X)$, and remark that by (4.2.1),

$$\mathcal{L}_u\phi(x) = \sum_{y, T_u y = x} e^{g(u,y)}\phi(y) \geq 0 \tag{4.5.4}$$

i.e $\mathcal{L}_u$ is a *positive* operator.
Therefore, by virtue of (4.5.3) and positivity of ϕ_u, ℓ_u is also positive on $C^\beta(X)$. It is then automatically continuous on $(C^\beta(X), \|.\|_{C^0})$: for every, $\phi \in C^\beta(X) \cup C^0_+(X)$, every $x \in X$, we have

$$-\|\phi\|_{C^0} \leq \phi(x) \leq \|\phi\|_{C^0} \tag{4.5.5}$$

so that

$$|\langle \ell_u, \phi\rangle| \leq \langle \ell_u, \mathbf{1}\rangle \|\phi\|_{C^0} \tag{4.5.6}$$

with $\langle \ell_u, \mathbf{1}\rangle = \langle \ell_u, \frac{1}{\phi_u}\phi_u\rangle = \int_X \frac{1}{\phi_u} dm_u > 0$.
Therefore, every $(\ell_u)_{u\in\mathcal{U}}$ is bounded on $(C^\beta(X), \|.\|_{C^0})$. From there, we can apply one of many classical extensions Theorem, such as the extension Theorem for uniformly continuous maps on a dense subset, to conclude that ℓ_u admits a bounded extension to $C^0(X)$. It naturally follows that m_u is a Radon measure.

For $s \in D(0, 1) \subset \mathbb{C}$, $u \in \mathcal{U}$ and $A \in C^{1+\alpha}(X)$, we introduce the parameter $\mathbf{u} = (s, u) \in D(0, 1) \times \mathcal{U} \subset \mathbb{C} \times \mathcal{B}$ and the weighted transfer operator (with weight e^g, $g : X \to \mathbb{R}$) $\mathcal{L}_{\mathbf{u}}$ defined on $C^{1+\alpha}(X)$ by

$$\mathcal{L}_{\mathbf{u}}\phi = \mathcal{L}_{s,u} = \mathcal{L}_u(e^{sA}\phi) \tag{4.5.7}$$

Note that $\mathcal{L}_{s,u}$ is an analytical perturbation of $\mathcal{L}_u$ (at a fixed $u \in \mathcal{U}$). Hence, $\mathcal{L}_{s,u}$ also has a spectral gap for $s \in D(0, r)$, with $r = r(u)$ small enough (cf. [47, IV, §3, Thm 3.12 et VII, § Thm 1.8]), and we will write $\lambda_{s,u}$, $\phi_{s,u}$ for its simple, maximal eigenvalue and the associated eigenvector (which is not necessarily a positive function, nor even a real valued one).
It follows from Ruelle Theorem 2.9 that $\lambda_{s,u} = e^{P(s,u)}$ with $P(s, u)$ the topological pressure associated with the dynamic T_u and the weight e^{sA+g}.
We now state a version of a well-known formula (cf. [68]), connecting topological pressure and the expectation of the observable A under the Gibbs measure m_u, suited to our needs.

Proposition 4.1
Let $u \in \mathcal{U}$. The map $s \in D(0, r_u) \mapsto P(s, u)$ is analytical and one has

$$\partial_s P(0, u) = m_u(A) \tag{4.5.8}$$

Proof. Fix $u \in \mathcal{U}$. For $s \in D(0,r)$, with $r = r(u)$ small enough, one can write $\mathcal{L}_{s,u}\phi_{s,u} = e^{P(s,u)}\phi_{s,u}$. The first statement follows from analytic perturbation theory, see [47, IV, §3, Thm 3.12 et VII, § Thm 1.8], as well as analyticity of $s \mapsto \ell_{s,u}$, with $\ell_{s,u}$ the eigenform for $\lambda_{s,u}$.

Furthermore, from the normalization $\langle \ell_{s,u}, \phi_{s,u} \rangle = 1$, one gets $\langle \ell_{s,u}, \mathcal{L}_{s,u}\phi_{s,u} \rangle = e^{P(s,u)}$ and by differentiating this last equality with respect to s, one has

$$\partial_s P(s,u)e^{P(s,u)} = \underbrace{\left(\langle \partial_s \ell(s,u), \phi_{s,u} \rangle + \langle \ell_{s,u}, \partial_s \phi_{s,u} \rangle\right)}_{(I)} e^{P(s,u)} + \underbrace{\langle \ell_{s,u}, \partial_s \mathcal{L}_{s,u}\phi_{s,u} \rangle}_{(II)} \tag{4.5.9}$$

On one hand, from $\langle \ell_{s,u}, \phi_{s,u} \rangle = 1$, one gets $(I) = 0$.
On the other hand $\partial_s \mathcal{L}_{s,u}\phi_{s,u} = \mathcal{L}_{s,u} A\phi_{s,u}$, so that we get $(II) = e^{P(s,u)}\langle \ell_{s,u}, A\phi_{s,u} \rangle$.
Finally, one has, at $s = 0$

$$\partial_s P(0,u) = \langle \ell_u, A\phi_u \rangle = m_u(A) \tag{4.5.8}$$

Fix a $u_0 \in \mathcal{U}$: thus $\lambda_{0,u_0} = \lambda_{u_0} > 0$.
One easily has, for all $y \in X$,

$$\mathcal{L}_{s,u}\phi(y) = \sum_{x \in T_u^{-1}y} e^{sA(x)+g(x)}\phi(x)$$

From Theorem 4.1, it holds that there is a neighborhood $D(0,r) \times B(u_0,\delta)$ such that $(s,u) \in D(0,r) \times B(u_0,\delta)$ implies $|\lambda_{s,u} - \lambda_{u_0}| \leq \frac{\lambda_{u_0}}{4}$.
In particular, r is independent of u and $\lambda_{s,u}$ is a positive real number. Hence $P(s,u)$ is correctly defined, and continuous with respect to $u \in B(u_0,\delta)$, for $s \in D(0,r)$.

From Theorem 4.2, it holds that there is a neighborhood $D(0,r') \times B(u_0,\delta')$ on which $(s,u) \longmapsto P(s,u)$ is C^1. In particular, $\partial_u P(s,u)$ exists and is continuous with respect to $u \in B(u_0,\delta')$ for $s \in D(0,r')$. Once again, r' is *a priori* independent of u.

From analytical perturbation theory, it holds that $s \in D(0,r") \longmapsto P(s,u)$ is analytical for $u \in B(u_0,\delta")$, where $r" = \min(r,r')$ and $\delta" = \min(\delta,\delta')$. Therefore, one can write, following Cauchy formula and (4.5.8)

$$m_u(A) = \int_{\mathcal{C}(0,r")} \frac{P(s,u)}{s^2} ds \tag{4.5.10}$$

where $\mathcal{C}(0,r")$ is the circle of radius $r"$ centered at 0.
By Lebesgue's Theorem, $u \in B(u_0,\delta") \longmapsto m_u(A)$ is a C^1 map. Up to a change in constants, this can be done for every $u_0 \in U$, thus concluding this proof. □

The last two points derive immediately from the general proposition, with the choice of weight $g_u = \dfrac{1}{|\det(DT_u)|}$ and $g_u = \mathbf{1}$. The linear response formula (4.5.2) follows from (4.1.2), the fact that for the weight $g_u = \dfrac{1}{|\det(DT_u)|}$, one has $\lambda_u = 1$, $\ell_u = \int_M .dm$ and the duality property (2.2.9). □

It is a classical fact (deriving, e.g, from the Nagaev-Guivarc'h spectral method, see [38]) that for a uniformly expanding, $C^{2+\alpha}$ map T_u, and Hölder observable $A : M \to \mathbb{R}$, the random processes $(A \circ T_u^n)_{n\geq 0}$ satisfies a central limit Theorem, with a centered limit distribution and variance $\Sigma_u^2(A)$ given by

$$\Sigma_u^2(A) := \int_M A^2 dm_u + 2\sum_{n=1}^{+\infty} \int_M A \circ T_u^n . A dm_u \tag{4.5.11}$$

with m_u the S.R.B measure, i.e the equilibrium state associated to the weight $\dfrac{1}{|\det(DT_u)|}$.

Corollary 4.3 (Regularity of the variance in the central limit Theorem)
We place ourselves in the same setting as in Theorem 4.2. Then the map $u \in \mathcal{U} \mapsto \Sigma_u^2$ *is* C^1.

Proof of Corollary 4.3 Placing ourselves in the same setting as in the proof of the last proposition, we work with the analytically perturbed Ruelle transfer operator $\mathcal{L}_{s,u} := \mathcal{L}_u(e^{sA}.)$ for observable $A \in C^{1+\alpha}(M)$ and small complex $s \in \mathbb{C}$, with an original weight $g_u := \frac{1}{\det(DT_u)}$. Still denoting by $P(s,u) = \log(\lambda_{s,u})$ the maximal eigenvalue of $\mathcal{L}_{s,u}$, we use the well-known formula

$$\partial_s^2 P(0,u) = \Sigma_u^2(A) \tag{4.5.12}$$

Indeed, starting from $\partial_s P(s,u) = \langle \ell_{s,u}, A\phi_{s,u}\rangle$ and differentiating with respect to s (which is licit by analytic perturbation theory), one obtains at $s = 0$.

$$\partial_s^2 P(0,u) = \langle \partial_s \ell_{0,u}, A\phi_u\rangle + \langle \ell_u, A\partial_s\phi_{0,u}\rangle \tag{4.5.13}$$

Now, recall that for any fixed $u \in \mathcal{U}$, by analytic perturbation theory[4] one has the following:

$$\partial_s\phi_{0,u} = (\mathbb{1} - \mathcal{L}_u)^{-1}\,\partial_s\mathcal{L}_{0,u}\phi_u \tag{4.5.14}$$
$$\partial_s\ell_{0,u} = (\mathbb{1} - \mathcal{L}_u^*)^{-1}\,\partial_s\mathcal{L}_{0,u}^*\ell_u \tag{4.5.15}$$

taking into account that $\ell_u = \int_M (.)dm$ and $\partial_s\mathcal{L}_{s,u} = \mathcal{L}_{s,u}(A.)$, we obtain

$$\partial_s\phi_{0,u} = (\mathbb{1} - \mathcal{L}_u)^{-1}\,\mathcal{L}_u[A\phi_u] \tag{4.5.16}$$
$$\partial_s\ell_{0,u} = (\mathbb{1} - \mathcal{L}_u^*)^{-1}\,A^*\mathcal{L}_u^*(dm) \tag{4.5.17}$$

so that (4.5.13) yields

$$\partial_s^2 P(0,u) = \int_M A\sum_{n=0}^{\infty}\mathcal{L}_u^n[A\phi_u]dm + \int_M A\sum_{n=1}^{\infty}\mathcal{L}_u^n[A\phi_u]dm \tag{4.5.18}$$
$$= \int_M A^2\phi_u dm + 2\sum_{n=1}^{\infty}\int_M A\circ T_u^n . A\phi_u dm \tag{4.5.19}$$

[4]The formula we obtain is similar to 4.1.2, but the fact that it is valid also for the left eigenvector $\ell_{s,u}$ is specific to the analytical case, where $s \mapsto \ell_{s,u}$ is analytic for small s

by using the duality property (2.2.9), boundedness of the linear form dm on Hölder spaces, together with the spectral gap property of $\mathcal{L}_u$.

From there, the conclusion follows from a Cauchy formula at order 2, similarly to the end of the proof of corollary 4.2. $\square$

4.6 Generalization to the discrete spectrum

For now, we were able to establish regularity result for the maximal eigenvalue of the transfer operator, and associated spectral data. In turn, this allowed us to study the regularity of some dynamical data. But not every dynamical data of interest is related to the top eigenvalue of the transfer operator: some are connected with other spectral data (e.g, the rate of mixing is bounded by the second largest eigenvalue [4, Prop 1.1]), which is why it is natural to investigate the regularity of the rest of the discrete spectrum.

Theorem 4.3
Let M be a smooth compact, connected, boundaryless Riemann manifold of finite dimension, $\mathcal{B}$ be a Banach space, and $u_0 \in \mathcal{B}$.
Let $0 \le \beta < \alpha$, and $(T_u)_{u\in\mathcal{U}}$ be a C^1 family of $C^{1+\alpha}$ expanding maps, and let $g \in C^{1+\alpha}(U \times M)$ be a (positive) function. Let $\mathcal{L}_{g,T_u} = \mathcal{L}_u$ be the weighted transfer operator, acting on the little Hölder space $c^{1+\alpha}(M)$ defined by (2.3.3).
Let $\lambda \notin \sigma(\mathcal{L}_{u_0})$, and $\mathcal{R}(\lambda, u_0) = (\lambda\mathbf{1} - \mathcal{L}_{u_0})^{-1}$ be the resolvent operator. Then there exists a (small enough) open neighborhood $\mathcal{U} \subset \mathcal{B}$ such that

- *For any $u \in \mathcal{U}$, the resolvent $\mathcal{R}(\lambda, u) \in L(c^{1+\alpha}(M))$ is well-defined. Furthermore, the map $u \in \mathcal{U} \mapsto \mathcal{R}(\lambda, u) \in L(c^{1+\alpha}(M), c^{1+\beta}(M))$ is $\gamma := \alpha - \beta$ Hölder.*
- *$u \in \mathcal{U} \mapsto \mathcal{R}(\lambda, u) \in L(c^{1+\beta}(M), c^{\beta}(M))$ is differentiable, and its differential satisfies*

$$\partial_u \mathcal{R}(\lambda, u) = -\mathcal{R}(\lambda, u)\partial_u \mathcal{L}_u \mathcal{R}(\lambda, u) \tag{4.6.1}$$

We want to deduce from Theorem 4.3 the regularity of the spectral projector Π_u on the eigenspace associated with λ_u. In order to achieve that goal, our strategy consists in applying Theorem 3.1, to a map constructed from the resolvent operator $\mathcal{R}(\lambda, u) = (\lambda\mathbf{1} - \mathcal{L}_u)^{-1}$.

Let $\lambda \notin \sigma(\mathcal{L}_{u_0})$ and $f \in c^r(M)$, $r \in \{\beta, \alpha, 1+\beta, 1+\alpha\}$. For every $u \in \mathcal{U}$, define the affine map $F : \mathcal{U} \times c^r(M) \mapsto c^r(M)$ by

$$F(u, \phi) := \frac{1}{\lambda} f - \frac{1}{\lambda}\mathcal{L}_u \phi \tag{4.6.2}$$

Given our choice of λ, there is a unique $\varphi_{u_0} \in c^r(M)$ such that $\lambda\varphi_{u_0} - \mathcal{L}_{u_0}\varphi_{u_0} = f$, i.e the map $F(u_0, .)$ admits $\varphi_{u_0} = \mathcal{R}(\lambda, u_0) f$ as a unique fixed point.

We will establish Theorem 4.3 by applying Theorem 3.1 to the map F. The first step consists in establishing both existence of $\mathcal{R}(\lambda, u) f \in c^{1+\alpha}(M)$ for $u \in \mathcal{U}$, and (Hölder) continuity of the map $u \in \mathcal{U} \mapsto \mathcal{R}(\lambda, u) f \in L(c^{1+\alpha}(M), c^{1+\beta(M)})$:

Proposition 4.2
Let M be a Riemann manifold, $\mathcal{B}$ be a Banach space, and $\mathcal{U} \subset \mathcal{B}$ be an open subset. Let $(T_u)_{u\in\mathcal{U}}$ be a Lipschitz family of $C^{1+\alpha}$ expanding maps, and let $g \in C^{1+\alpha}(U \times M)$ be a (positive) function. Let $\mathcal{L}_{g,T_u} = \mathcal{L}_u$ be the weighted transfer operator defined by (4.2.1), acting on $c^{1+\alpha}(M)$.
Let $u_0 \in \mathcal{U}$ and $\lambda \notin \sigma(\mathcal{L}_{u_0})$, and $\mathcal{R}(\lambda, u_0) = (\lambda\mathbf{1} - \mathcal{L}_{u_0})^{-1}$ be the resolvent operator. Let $0 \leq \beta < \alpha$, and $\gamma := \alpha - \beta$.

Then there is a neighborhood $u_0 \in \mathcal{U}' \subset \mathcal{U}$ such that $\mathcal{R}(\lambda, u) \in L(c^{1+\alpha}(M))$ is well defined for every $u \in \mathcal{U}'$, and the map $u \in \mathcal{U} \longmapsto \mathcal{R}(\lambda, u) \in L(c^{1+\alpha}(M), c^{1+\beta}(M))$ is γ-Hölder.

Proof. This proposition is mostly a consequence of the *Keller-Liverani Theorem* 2.10, whose assumptions have been established at various places in this text (uniform Lasota-Yorke inequalities in Theorem 2.9, continuity in relative topology in §4.3).

Let $\lambda \notin \sigma(\mathcal{L}_{u_0})$ and $u, v \in \mathcal{U}'$. Then by Theorem 2.10, $\mathcal{R}(\lambda, u)$ and $\mathcal{R}(\lambda, v)$ are well-defined on $c^{1+\alpha}(M)$. We have this easy variant of the famous resolvent formula :

$$\underbrace{\mathcal{R}(\lambda, u) - \mathcal{R}(\lambda, v)}_{\in L(c^{1+\alpha}(M), c^{1+\beta}(M))} = \underbrace{(\lambda\mathbf{1} - \mathcal{L}_u)^{-1}}_{\in L(c^{1+\beta}(M))} \underbrace{(\mathcal{L}_u - \mathcal{L}_v)}_{\in L(c^{1+\alpha}(M), c^{1+\beta}(M))} \underbrace{(\lambda\mathbf{1} - \mathcal{L}_v)^{-1}}_{\in L(c^{1+\alpha}(M))} \tag{4.6.3}$$

This formula allows one to reduce the problem of regularity for $u \in \mathcal{U} \longmapsto \mathcal{R}(\lambda, u) \in L(c^{1+\alpha}(M), c^{1+\beta}(M))$ to the regularity of $u \in \mathcal{U} \longmapsto \mathcal{L}_u \in L(C^{1+\alpha}(M), C^{1+\beta}(M))$, which was established in lemma 4.1.

$$||\mathcal{L}_u - \mathcal{L}_v||_{C^{1+\alpha}, C^{1+\beta}} \leq C(u, v)||u - v||^\gamma \tag{4.6.4}$$

with $C(u, v)$ a constant, depending on the Lipschitz and Hölder norms of the inverse branches of the dynamic, which can be chosen uniform in (u, v) if they lie in a (small enough) neighborhood of a $u_0 \in \mathcal{U}$ (cf. the proof of lemma 4.1, also in [73, Lemma 5, section 3.2]).
Interpreting $\mathcal{R}(\lambda, v)$ as an operator on $C^{1+\alpha}$, $\mathcal{L}_u - \mathcal{L}_v$ as an operator from $C^{1+\alpha}$ to $C^{1+\beta}$, and $\mathcal{R}(\lambda, u)$ as an operator on $C^{1+\beta}$, one gets for every $f \in C^{1+\alpha}$, every $u, v \in \mathcal{U}$

$$\begin{aligned} ||(\mathcal{R}(\lambda, u) - \mathcal{R}(\lambda, v))f||_{C^{1+\beta}} &\leq ||\mathcal{R}(\lambda, u)||_{C^{1+\beta}}||||\mathcal{L}_u - \mathcal{L}_v||_{C^{1+\alpha}, C^{1+\beta}}||\mathcal{R}(\lambda, v)f||_{C^{1+\alpha}} \\ &\leq C(u, v)||\mathcal{R}(\lambda, u)||_{C^{1+\beta}}||\mathcal{R}(\lambda, v)||_{C^{1+\alpha}}||f||_{C^{1+\alpha}}||u - v||^\gamma \end{aligned}$$

Under those conditions, one gets

$$||(\mathcal{R}(\lambda, u) - \mathcal{R}(\lambda, v))||_{C^{1+\alpha}, C^{1+\beta}} \leq C(u_0)||\mathcal{R}(\lambda, u)||_{C^{1+\beta}}||\mathcal{R}(\lambda, v)||_{C^{1+\alpha}}||u - v||^\gamma \tag{4.6.5}$$

so that $u \in \mathcal{U} \mapsto \mathcal{R}(\lambda, u) \in L(c^{1+\alpha}(M), c^{1+\beta}(M))$ is (locally) Hölder, and therefore continuous at every $u_0 \in \mathcal{U}$, if one can get *uniform* (in u) bounds on the $C^{1+\alpha}$, $C^{1+\beta}$ norms of $\mathcal{R}(\lambda, u)$. This last fact follows from Keller-Liverani Theorem 2.10. We can now conclude with (2.5.4) that

$$\|(\mathcal{R}(\lambda, u) - \mathcal{R}(\lambda, v))\|_{C^{1+\alpha}, C^{1+\beta}} \leq C(u_0)\|u - v\|^\gamma \tag{4.6.6}$$

It is noteworthy that we obtain *uniform bounds* on the resolvent, both in u and in f, i.e that we obtain Hölder continuity for the resolvent in the operator topology. □

The second step of the proof of Theorem 4.3 consists in establishing that $(u, \phi) \in \mathcal{U} \times c^{1+\alpha}(M) \mapsto \frac{1}{\lambda}\mathcal{L}_u\phi + \frac{1}{\lambda}f$ satisfies (3.2.2). But λ, f are independent of (u, ϕ), so that it is enough to establish (3.2.2) for $(u, \phi) \in \mathcal{U} \times c^{1+\alpha}(M) \mapsto \mathcal{L}_u\phi \in c^\beta(M)$. This is what was done in lemma 4.1, in the space $C^\beta(M)$. It extends unchanged to the present setting, as the norms used are the same.

The last step of the proof consists in showing the invertibility (and boundedness of the inverse) for the partial differential (w.r.t ϕ) of the map $(u, \phi) \in \mathcal{U} \times c^{1+\alpha}(M) \mapsto \frac{1}{\lambda}\mathcal{L}_u\phi + \frac{1}{\lambda}f$ in the space $c^\beta(M)$. But it is trivial to see that this partial differential is just $Id - \frac{1}{\lambda}\mathcal{L}_u$, so that its invertibility on c^β boils down to the invariance of peripheral spectrum for $\mathcal{L}_u$ when changing spaces to $c^\beta(M)$ from $c^{1+\alpha}(M)$. This last fact is a consequence of the following result (its proof can be found in [6, Appendix A, p.212, lemma A.3])

Lemma 4.2
Let $\mathcal{B}$ be a separated topological linear space, and let $(\mathcal{B}_1, ||.||_1)$ and $(\mathcal{B}_2, ||.||_2)$ be Banach spaces, continuously embedded in $\mathcal{B}$. Suppose there is a subspace $\mathcal{B}_0 \subset \mathcal{B}_1 \cap \mathcal{B}_2$ that is dense in both $(\mathcal{B}_1, ||.||_1)$, $(\mathcal{B}_2, ||.||_2)$. Let $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ be a linear continuous map, preserving $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, and such that its restriction to $\mathcal{B}_1$, $\mathcal{B}_2$ are bounded operators, whose essential spectral radii are both strictly smaller than some $\rho > 0$. Then the eigenvalues of $\mathcal{L}|_{\mathcal{B}_1}$ and $\mathcal{L}|_{\mathcal{B}_2}$ on $\{z \in \mathbb{C}, \ |z| > \rho\}$ coincide. Furthermore, the corresponding generalized eigenspaces coincide and are contained in $\mathcal{B}_1 \cap \mathcal{B}_2$.

This allows us to conclude that for every $f \in c^{1+\alpha}(M)$, the map $u \in \mathcal{U} \mapsto \mathcal{R}(\lambda, u)f \in c^\beta(M)$ is differentiable, i.e point-wise differentiability of the resolvent operator viewed as an element of $L(c^{1+\alpha}(M), c^\beta(M))$.

A natural question is therefore the possibility to extend that result in the operator topology, i.e to show differentiability of $u \in \mathcal{U} \mapsto \mathcal{R}(\lambda, u) \in L(c^{1+\alpha}(M), c^\beta(M))$. Writing (3.2.2) for the map F at $(u_0, \mathcal{R}(\lambda, u_0)f)$, one gets

$$\begin{aligned}&[\mathcal{R}(\lambda, u_0 + h) - \mathcal{R}(\lambda, u_0) + \mathcal{R}(\lambda, u_0)P_{u_0}\mathcal{R}(\lambda, u_0)]\, f \\ &= \epsilon(\|h\|_{\mathcal{B}}, \|\mathcal{R}(\lambda, u_0 + h)f - \mathcal{R}(\lambda, u_0)f\|_{C^{1+\beta}})\, [\|h\|_{\mathcal{B}} + \|\left(\mathcal{R}(\lambda, u_0 + h) - \mathcal{R}(\lambda, u_0)\right) f\|_{C^\beta}]\end{aligned} \tag{4.6.7}$$

Looking carefully at the error term $\epsilon(\|h\|_{\mathcal{B}}, \|\mathcal{R}(\lambda, u_0 + h)f - \mathcal{R}(\lambda, u_0)f\|_{C^{1+\beta}})$, we see that it is bounded uniformly in $\|f\|_{C^{1+\alpha}}$ (take $z = \mathcal{R}(\lambda, u_0 + h)f - \mathcal{R}(\lambda, u_0)f$ in (4.2.12)). Therefore, we obtain

$$\sup_{\|f\|_{C^{1+\alpha}} \leq 1} \|\, [\mathcal{R}(\lambda, u_0 + h) - \mathcal{R}(\lambda, u_0) + \mathcal{R}(\lambda, u_0)P_{u_0}\mathcal{R}(\lambda, u_0)]\, f\|_{C^\beta} = \|h\|_{\mathcal{B}}\epsilon(h) \tag{4.6.8}$$

i.e $u \mapsto \mathcal{R}(\lambda, u) \in L(c^{1+\alpha}(M), c^\beta(M))$ is differentiable at $u_0 \in \mathcal{U}$. □

The final part of this note is to deduce regularity of the spectral projector map $u \in \mathcal{U} \mapsto \Pi_u \in L(c^{1+\alpha}(M), c^\beta(M))$.

Theorem 4.4
Let M be a smooth compact, connected, boundaryless Riemann manifold of finite dimension, $\mathcal{B}$ be a Banach space, and $\mathcal{U} \subset \mathcal{B}$ be an open subset. Let $0 \leq \beta < \alpha$, and $(T_u)_{u \in \mathcal{U}}$ be a C^1 family

of $C^{1+\alpha}$ expanding maps, and let $g \in C^{1+\alpha}(U \times M)$ be a (positive) function. Let $\mathcal{L}_{g,T_u} = \mathcal{L}_u$ be the weighted transfer operator, acting on the little Hölder space $c^{1+\alpha}(M)$. *Let λ_u be a discrete eigenvalue of finite multiplicity, with $|\lambda_u| > \rho_{ess}(\mathcal{L}_u)$, and Π_u be the spectral projector on the associated generalized eigenspace. Then we have the following :*

- *The map $u \in \mathcal{U} \mapsto \Pi_u \in L(c^{1+\alpha}(M), c^{1+\beta}(M))$ is $\gamma := \alpha - \beta$ Hölder.*
- *$u \in \mathcal{U} \mapsto \Pi_u \in L(c^{1+\beta}(M), c^{\beta}(M))$ is differentiable.*

Proof.
Recall the formula defining a spectral projector: denoting by λ an isolated eigenvalue of a bounded operator $\mathcal{L}$, and letting $\mathcal{C}$ be a circle centered at λ, small enough not to encounter the rest of the spectrum, one has

$$\Pi = \frac{1}{2i\pi}\int_{\mathcal{C}}(z-\mathcal{L})^{-1}dz$$

.

In our setting, this reads $\Pi_u = \frac{1}{2i\pi}\int_{\mathcal{C}_u}\mathcal{R}(\lambda,u)d\lambda$, with $\mathcal{C}_u$ any closed curve encircling λ_u (and no other element of $\mathcal{L}_u$'s spectrum). The most natural idea now is to use a "regularity under the integral" result.

The first step is to pick a closed curve that is independent of $u \in \mathcal{U}$, at least in a neighborhood of a $u_0 \in \mathcal{U}$. Let $u_0 \in \mathcal{U}$, and take u in a (small enough) neighborhood U' of u_0. It follows from Keller-Liverani Theorem 2.10 that whenever $u \in U' \mapsto \mathcal{L}_u \in L(c^{1+\alpha}(M), c^{1+\beta}(M))$ is (uniformly) continuous, then any finite set of isolated points $\lambda_1(u), .., \lambda_k(u)$ in $\mathcal{L}_u$'s discrete spectrum also depends (uniformly) continuously on $u \in U'$. Therefore one can pick a radius r_{u_0} such that for every $u \in U'$, $\lambda_u \in D(\lambda_{u_0}, \frac{r_{u_0}}{2})$ and there is no other element of the spectrum of $(\mathcal{L}_u)_{u\in U'}$ in this disk. It is then natural to pick $\mathcal{C}_{u_0} = \mathcal{C}(\lambda_{u_0}, \frac{r_{u_0}}{2})$ the circle centered in λ_{u_0} of radius $\frac{r_{u_0}}{2}$. Hence one can write, for any $u \in U'$:

$$\Pi_u = \frac{1}{2i\pi}\int_{\mathcal{C}_{u_0}}\mathcal{R}(\lambda,u)d\lambda \tag{4.6.9}$$

The second step is to bound $\partial_u\mathcal{R}(\lambda,u) \in L(c^{1+\alpha}(M), c^{\beta}(M))$ independently of $u \in U'$.
Up to reduce further U', one has the formula $\partial_u\mathcal{R}(\lambda,u)f = -\mathcal{R}(\lambda,u)\partial_u\mathcal{L}_u\mathcal{R}(\lambda,u)f$ for $u \in U'$. Through a standard trick, one can show that $\partial_u\mathcal{R}(\lambda,u)f - \partial_u\mathcal{R}(\lambda,u_0)f$ is a sum of terms involving only products of the terms $(\mathcal{R}(\lambda,u)-\mathcal{R}(\lambda,u_0))$, $(\partial_u\mathcal{L}_u - \partial_u\mathcal{L}_{u_0})$, $\mathcal{R}(\lambda,u_0)$ and $\partial_u\mathcal{L}_{u_0}$. Thus it is enough to bound the first two above (independently of $u \in U'$). Bounding the first expression, one gets $||\mathcal{R}(\lambda,u)-\mathcal{R}(\lambda,u_0)||_{C^{1+\alpha},C^{\beta}} \leq C(u_0,\lambda)diam(U')^{1+\gamma}$, thanks to estimate (3.1.18) in [73, p.13, lemma 4].
To bound $(\partial_u\mathcal{L}_u - \partial_u\mathcal{L}_{u_0})$ in C^{β}-norm, note that one has the formula:

$$\partial_u\mathcal{L}_u\phi = -\mathcal{L}_u[\phi\frac{dg}{g}.M_u + d\phi.M_u] \tag{4.6.10}$$

with $M_u(x) = (dT_u(x))^{-1} \circ \partial_u T_u(x) : \mathcal{B} \to L(T_{T_u x}M, T_x M)$ is linear and bounded for every $x \in M$. Therefore it is enough to establish that :

(1) $||\mathcal{L}_u - \mathcal{L}_{u_0}||_{C^{1+\alpha},C^{\beta}}$ is bounded independently of $u \in U'$.

(2) M_u can be bounded independently of $u \in U'$

The first item is just another avatar of estimate (3.1.18) in [73, p.13, lemma 4]. The second naturally follows from the regularity hypothesis made on $u \in \mathcal{U} \mapsto T_u$, as one has immediately: $||M_u(x)|| \leq \sup_{u\in\mathcal{U}} ||DT_u^{-1}||_{L(T_{T_u x}M, T_x M)} ||\partial_u T_u(x)||_{L(\mathcal{B}, T_{T_u x}M)}$.

One then gets a bound for $||\partial_u \mathcal{R}(\lambda, u)||_{C^\beta}$, independent of u in U', and trivially integrable in λ on $\mathcal{C}_{u_0}$. Therefore, one can apply the Lebesgue's Theorem for differentiability under the integral, and get the announced result. $\square$

Chapter 5

Top characteristic exponent of cocycles of transfer operators : perturbations and dynamical applications

5.1 Introduction

5.1.1 The problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\tau : \Omega \to \Omega$ a measure-preserving, invertible, and ergodic map. Let $\mathcal{B}$ be a Banach space and $\mathcal{U} \subset \mathcal{B}$ an open subset. Let M be a compact, connected Riemann manifold, and let $\mathbb{M} = M \times \Omega$ be a (trivial) bundle over Ω, with $M_\omega = M \times \{\omega\}$ the fiber above $\omega \in \Omega$.

For $\omega \in \Omega$, $u \in \mathcal{U}$, and $r > 1$, we assume that $T_{\omega,u} : M_\omega \to M_{\tau\omega}$ is a uniformly expanding C^r map (see (5.2.20)-(5.2.22) for precise conditions), and we consider the random product of expanding maps above (Ω, τ) defined by

$$T^{(n)}_{\omega,u} := T_{\tau^{n-1}\omega,u} \circ \cdots \circ T_{\omega,u} \tag{5.1.1}$$

If $\boldsymbol{g}_u = (g_{\omega,u})_{\omega\in\Omega} \in L^\infty(\Omega, C^{r-1}(M))$, we let $\mathcal{L}_{\omega,u} : E^s_\omega = C^s(M_\omega) \to E^s_{\tau\omega}$ be the (weighted) transfer operator associated with $T_{\omega,u}$, acting on $\phi \in C^s(M)$ by

$$\mathcal{L}_{\omega,u}\phi(y) := \sum_{x, T_{\omega,u}x=y} g_{\omega,u}(x)\phi(x) \tag{5.1.2}$$

for every $y \in M$. This transfer operator generates a random cocycle $\mathcal{L}^{(n)}_{\omega,u}$ by taking the product along the τ orbit:

$$\mathcal{L}^{(n)}_{\omega,u} = \mathcal{L}_{\tau^{n-1}\omega,u} \circ .. \circ \mathcal{L}_{\omega,u} \tag{5.1.3}$$

Under the additional assumption that $\log \|\mathcal{L}_{\omega,u}\|_{C^s} \in L^1(\Omega, \mathbb{P})$ (which comes naturally from uniform boundedness assumptions on the weights $g_{\omega,u}$), Kingman's theorem 2.5 insures the existence of the **top characteristic exponent** of the cocycle, defined for every $u \in \mathcal{U}$ by[1]

$$\chi_{\omega,u} = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^{(n)}_{\omega,u}\|_s \tag{5.1.4}$$

By the ergodicity assumption on τ, one has that $\chi_{\omega,u} = \chi_u$ is $\mathbb{P}$ almost surely independent of $\omega \in \Omega$.
One of the goals of this chapter is to study the map $u \in \mathcal{U} \longmapsto \chi_u$ regularity.

Throughout this chapter, we will consider measurability of maps from Ω to a Banach space $\mathcal{B}$ (e.g Hölder spaces $(C^r(M))_{r>0}$) in the sense of Bochner, i.e that each measurable function $\phi : \Omega \to \mathcal{B}$ is a (point-wise) limit of simple (i.e linear combination of indicator functions) maps. If the Banach space $\mathcal{B}$ is separable (which is the case for Hölder spaces $C^r(M)$ if and only if $r \in \mathbb{N}$), Bochner measurability coincides with standard measurability.

It is of major interest that the top characteristic exponent of (5.1.3) (and more generally its Oseledets-Lyapunov spectrum) plays a key role in the study of the dynamical properties of random products of maps, similar in many aspects to the one played by the spectrum of the transfer operator in the study of statistical properties of autonomous dynamical systems.
If this general philosophy was already driving the first studies of random product of matrices with the seminal papers of Oseledets, [60], this approach was first applied to cocycles of transfer operators by Froyland&al (see [28, 29, 35]).

However, the problem of (parameter-wise) regularity for the top characteristic exponent has been widely investigated, both in the framework of cone contraction and in a more general setting: starting from Ruelle seminal paper [62], to Le Page [54], establishing Hölder and smooth regularity in the case of an i.i.d product of matrices, as well as Hennion's paper [44]. More recent works on Lyapunov exponents stability, formulated in the framework of transfer operator cocycles, include [26, 27, 19]. One may also mention [16], where the genericity of analyticity of Lyapunov exponents for random bounded linear operators is shown. This investigation on characteristic exponent regularity w.r.t parameters was also recently used to establish a random analogue of the Nagaev-Guivarc'h method (see [18]).
In the setting of cone contractions, one can think of Dubois article on analytical regularity of the top characteristic exponent for an analytically perturbed cone contracting operator [20], or to Rugh's paper [71] generalizing Ruelle and Dubois results to the framework of $\mathbb{C}$-cones.

Another goal of this chapter is to show how one can use (complex) cone contraction theory to establish the existence of a stationary measure for a random product of expanding maps on M, and to study its parameter dependency. In fact, we will see that we can construct an absolutely continuous invariant probability measure, and that it is intimately related to the Oseledets space associated with the top characteristic exponent. The question of its dependence w.r.t parameters is thus a natural extension of our previous one.

[1] It is *a priori* not clear whether χ is independent of the norm $\|.\|_s$, or that it is finite (i.e not $-\infty$). We will see that it is in fact the case.

The approach of using cone contraction to construct (quenched) stationary measures for random product of expanding maps, and to study their dependencies on parameters is not new: e.g, it is used in the study of stochastic stability and random correlations for a random product of expanding maps in [7].
More recently, there was a surge of interest in the question of linear response for random dynamical systems, with the recent preprints [1, 30], and the question of regularity and response of stationary measures in the context of random dynamics is certainly of major interest in many applications, notably for climate science: see the seminal paper by Hairer and Majda [42], Lucarini's work [58], or the surveys by Ghil&al [33, 13].

5.1.2 Main result and outline of the strategy

It is well known ([55],[4, Chap.2]) that for a C^r, $r > 1$ expanding dynamic T, acting on a Riemann manifold M, one can construct the unique invariant density in the following way: the associated transfer operator $\mathcal{L}_T$ is a *cone contraction*, i.e there exists a Birkhoff cone $\mathcal{C}$ (see def.A.2) in the functional space on which $\mathcal{L}$ acts (e.g, $C^{r-1}(M)$) that is strictly mapped on its interior (cf §5.2). When $\mathcal{C}$ is endowed with the proper projective metric (the so-called Hilbert projective metric, see definition A.3), this makes $\mathcal{L}$ a (strict and uniform) contraction of a Cauchy space, therefore allowing one to use the Banach contraction principle to construct a "fixed point" in the projective space (i.e an eigenfunction for $\mathcal{L}$). Properly normalized, this fixed point is actually the invariant density of the dynamical system T.

Here instead of iterating a fixed map, at each step we choose at random an expanding map (with uniform conditions on the dilation constant), and then perturb it in a smooth way. In this context, can one construct an invariant density ?
At the level of transfer operators, it is akin to fixing a cone $\mathcal{C}$ with good geometric properties (namely that it is a *regular Birkhoff cone*) and choosing at random an operator contracting the cone $\mathcal{C}$, perturb in a (unfortunately) non-smooth way the cocycle it generates. Can we still construct a generalized eigenvector for the transfer operator cocycle ? If so, how does this generalized eigenvector respond to the perturbation ? Can we still give a dynamical interpretation of this construction ?

Before formulating precisely our answers, we need to introduce a few notations and specify a few things: in this chapter, the objects we will consider have a dual interpretation: first as a "fiber-wise" object, and second as a "global", bundle object. We will denote in simple characters the first and in bold characters the second. For example, each family $(\phi_\omega)_{\omega\in\Omega} \in C^r(M)$ with $\operatorname{ess\,sup}_{\omega\in\Omega} \|\phi_\omega\|_{C^r(M)} < +\infty$ induces a map $\boldsymbol{\phi} \in L^\infty(\Omega, C^r(M))$. In the same spirit, under the assumption $\operatorname{ess\,sup}_{\omega\in\Omega} \|\mathcal{L}_\omega\|_{C^r(M)} < +\infty$ the cocycle of transfer operator $\mathcal{L}_\omega \in L(C^r(M_\omega), C^r(M_{\tau\omega})$, induces a bundle operator $\boldsymbol{\mathcal{L}} \in L(L^\infty(\Omega, C^r(M)))$, acting on $\boldsymbol{\phi} \in L^\infty(\Omega, C^r(M))$ by

$$(\boldsymbol{\mathcal{L}}\boldsymbol{\phi})_\omega := \mathcal{L}_{\tau^{-1}\omega}\phi_{\tau^{-1}\omega}$$

Our strategy is to exploit the regularity condition on some well-chosen cone $\mathcal{C}$ of a Banach space E. This property allows us to consider $\ell \in E^*$ a linear form, non zero on $\mathcal{C}^*$ (see Appendix A), so that we can consider $\boldsymbol{\pi}$, the projection on the affine hyperplane $\{\ell = 1\}$, defined by

$$(\boldsymbol{\pi}\boldsymbol{\phi})_\omega = \frac{\mathcal{L}_{\tau^{-1}\omega}\phi_{\tau^{-1}\omega}}{\langle \ell, \mathcal{L}_{\tau^{-1}\omega}\phi_{\tau^{-1}\omega}\rangle} \tag{5.1.5}$$

for $\phi \in L^\infty(\Omega, \mathcal{C} \cap \{\ell = 1\})$.

It is shown in appendix A that this projector is indeed a contraction of $L^\infty(\Omega, \mathcal{C}\cap\{\ell = 1\})$, so that one can construct its (unique) fixed point $\mathbf{h} : \Omega \to E$. This map is measurable and bounded. Furthermore, for a cocycle of transfer operators, this fixed point is the generalized eigenvector we alluded to earlier, i.e

$$\mathcal{L}_{\tau^{-1}\omega} h_{\tau^{-1}\omega} = h_\omega$$

as well as the (random) invariant density of the random product (see the beginning of section 5.2 and Proposition 5.1).

In what follows, we give abstract theorems that presents our results on parameter dependency of this fixed point. Given a cone $\mathcal{C}$ in a Banach space E, we will be interested in $\mathcal{M}_C(\Delta, \rho)$ the set of all bounded operators $\mathcal{L} : E \to E$ who are strict contractions of $\mathcal{C}$ with uniform control on the size of $\mathcal{L}(\mathcal{C})$ (see Definition A.8 for the precise definition).

Definition 5.1
Let $r > 0$, and $(\mathcal{B}_s)_{s\in(0,r]}$ be Banach spaces, such that for $0 \le s \le s'$, $\mathcal{B}_{s'} \subset \mathcal{B}_s$ and the injection is a bounded operator (which we will denote $B_{s'} \hookrightarrow \mathcal{B}_s$). We call such a family $(\mathcal{B}_s)_{s\in(0,r]}$ of Banach spaces a scale of Banach spaces*.*

Theorem 5.1
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\tau : \Omega \to \Omega$ be a measure-preserving, $\mathbb{P}$-ergodic, invertible map.
Let $\mathcal{B}$, $\mathcal{B}_0 \overset{\tilde{\jmath}}{\hookleftarrow} \mathcal{B}_1$, $\mathcal{U} \subset \mathcal{B}$ be an open subset, and for $u \in \mathcal{U}$ we let $\boldsymbol{\mathcal{L}}_u$ be a family of cocycles above (Ω, τ) such that:

(i) For every $u \in \mathcal{U}$, for every $s \in \{0, 1\}$, $\boldsymbol{\mathcal{L}}_u \in L^\infty(\Omega, L(\mathcal{B}_s))$

(ii) For every $u \in \mathcal{U}$, every $s \in \{0, 1\}$, $\log \|\boldsymbol{\mathcal{L}}_u\|_{\mathcal{B}_s} \in L^1(\Omega, \mathbb{P})$.

(iii) The map $u \in \mathcal{U} \mapsto \boldsymbol{\mathcal{L}}_u \in L^\infty(\Omega, L(\mathcal{B}_1, \mathcal{B}_0))$ is Lipschitz.

(iv) Let $u_0 \in \mathcal{U}$, and $s \in \{0, 1\}$. There exists a regular Birkhoff cone $\mathcal{C}_s \subset \mathcal{B}_s$ such that $\boldsymbol{\mathcal{L}}_u \in \mathcal{M}_{\mathcal{C}_s}(\Delta, \rho)$ (i.e $\mathcal{L}_{\omega,u}$ is a strict and uniform contraction of the cone $\mathcal{C}_s$) for every $u \in \mathcal{U}$ close enough to u_0.
Furthermore, we assume that $\tilde{\jmath}(\mathcal{C}_1) \subset \mathcal{C}_0$, and that the outer regularity form ℓ_0 extend ℓ_1, i.e that $\ell_0 \circ \tilde{\jmath} = \ell_1$.

Then the fixed point map $u \in \mathcal{U} \mapsto \boldsymbol{h}_u \in L^\infty(\Omega, \mathcal{B}_0)$ is Lipschitz. Furthermore, the top characteristic exponent χ_u is independent of the chosen norm, and is a Lipschitz map of the parameter.

It is natural to ask for higher-order regularity results concerning the fixed point map $u \in \mathcal{U} \mapsto \boldsymbol{h}_u$ and characteristic exponent $u \in \mathcal{U} \mapsto \chi_u$. Following chapter 3 (or [73, §2.3]), we will use the notion of **graded calculus**. More precisely, we can show the following:

Theorem 5.2
We make the same assumptions as in theorem 5.1. Let $\mathcal{B}_0 \hookleftarrow \mathcal{B}_1 \hookleftarrow \ldots \hookleftarrow \mathcal{B}_r$. Furthermore, we assume that

(iv) The map $(u,\phi)\in\mathcal{U}\times\mathcal{B}_r\mapsto\boldsymbol{\mathcal{L}}_u\phi\in L^\infty(\Omega,\mathcal{B}_0)$ admits the following "Taylor expansion":

There exist random operators $(\boldsymbol{Q}^{(i,j)})_{i+j\le r}$, $\boldsymbol{Q}^{(i,j)}\in L^\infty(\Omega,L(\mathcal{B}^i\times\mathcal{B}^j_{r-(i+j)},\mathcal{B}_0))$, and for every $u\in\mathcal{U}$, every $v\in\mathcal{B}$ for which $u+v\in\mathcal{B}$, every $(\phi_r,z_r)\in\mathcal{B}_r$, one has

$$\boldsymbol{\mathcal{L}}(u+v,\phi_0+z_0)-\boldsymbol{\mathcal{L}}(u,\phi_0)=\sum_{p=1}^{r}\sum_{\substack{(i,j)\\ i+j=p}}\boldsymbol{Q}^{(i,j)}(u,\phi_p)\,[v,z_{p-1}]+\boldsymbol{\mathcal{R}}(v,z_r)\tag{5.1.6}$$

where $\boldsymbol{\mathcal{R}}(h,z_r)=[\|h\|^r_{\mathcal{B}}+\|z_0\|^r_0]\,\epsilon(v,z_r)$ and $\epsilon(v,z_r)\underset{\substack{v\to0\\ z_r\to0}}{\longrightarrow}0$ in the $L^\infty(\Omega,\mathcal{B}_0)$ topology.

Then the fixed point map $\boldsymbol{h}_{0,u}\in L^\infty(\Omega,\mathcal{B}_0)$, and the top characteristic exponent χ_u of $\boldsymbol{\mathcal{L}}_u$ are r-1 times differentiable with respect to $u\in\mathcal{U}$.
Furthermore, one has the following Taylor expansion[2]:

$$\begin{aligned}\boldsymbol{h}_{u+v}-\boldsymbol{h}_u&=\left[\mathbb{1}-\boldsymbol{Q}^{(0,1)}_{\boldsymbol{\pi}}(u,\boldsymbol{h}_u)\right]^{-1}\boldsymbol{Q}^{(1,0)}_{\boldsymbol{\pi}}(u,\boldsymbol{h}_u).v\\&+\left[\mathbb{1}-\boldsymbol{Q}^{(0,1)}_{\boldsymbol{\pi}}(u,\boldsymbol{h}_u)\right]^{-1}\sum_{i+j=2}\boldsymbol{Q}^{(i,j)}_{\boldsymbol{\pi}}(u,\boldsymbol{h}_u)[v,D_u\boldsymbol{h}_u.v]+o(\|v\|^2_{\mathcal{B}})\end{aligned}\tag{5.1.7}$$

$$\begin{aligned}\chi_{u+v}-\chi_u&=\int_\Omega\frac{1}{\boldsymbol{p}_u}\langle\ell_0,\boldsymbol{Q}^{(1,0)}_{\mathcal{L}}.v+\boldsymbol{\mathcal{L}}.[(\mathbb{1}-\boldsymbol{Q}^{(0,1)}_{\boldsymbol{\pi}}(u,\boldsymbol{h}_u))^{-1}\boldsymbol{Q}^{(1,0)}_{\boldsymbol{\pi}}(u,\boldsymbol{h}_u)].v\rangle d\mathbb{P}\\&+\int_\Omega\frac{1}{\boldsymbol{p}_u}\langle\ell_0,\sum_{\substack{(i,j)\\ i+j=2}}\boldsymbol{Q}^{(i,j)}[v,D_u\boldsymbol{h}_u.v]+\boldsymbol{Q}^{(0,1)}[D^2_u\boldsymbol{h}_u.v]\rangle d\mathbb{P}+o(\|v\|^2_{\mathcal{B}})\end{aligned}\tag{5.1.8}$$

It is possible to go even further, and to prove Taylor expansions for the random a.c.i.m and the top characteristic exponent at every order. But to quote Gouëzel [37], "the computations are straightforward, but the notations are awful". Therefore, in order to keep sane both our reader and the author, we will stop at order 2.

The chapter is organized as follows: the abstract theorems 5.1 and 5.2 are proven in §5.3. In §5.2, we exhibit a family of regular Birkhoff cones in C^k spaces that are contracted in a strict and uniform way by the transfer operators of a broad class of expanding maps. In turns, it allows us to apply theorems 5.1 and 5.2, and to derive a variety of consequence: quenched and annealed linear response for the random physical measure (theorems 5.4 and 5.5), regularity of the variance in the central limit theorem satisfied by a random product of expanding maps (theorem 5.3), or of the Hausdorff dimension of the repeller associated to a random product of one-dimensional expanding maps (so-called "cookie-cutters").
In appendix A material on (Birkhoff and complex) cone contraction, as well as random products of such operators is collected. Technical lemmas on regularity of random composition operators on Hölder spaces are presented in appendix B.

5.2 Dynamical applications

In this section, we will work with C^r uniformly expanding maps on M and Hölder spaces $(C^s(M))_{0<s\le r}$. Our strategy is to study the fixed point $\boldsymbol{h}_u\in L^\infty(\Omega,C^s(\mathbb{M}))$ to study regularity of both the stationary measure and top characteristic exponent, by an extensive use of

[2]The formula (4.2.8) should be understood in $\mathcal{B}_0$. We omitted the indices on $\boldsymbol{h}$ to keep the formula readable.

theorem 3.1.
To apply this theorem and establish regularity for the map $u \in \mathcal{U} \mapsto \mathbf{h}_u$, we will need the following ingredients:

- Existence of a family of regular $\mathbb{R}$-cones in $C^s(M)$, stable by the transfer operators $\mathcal{L}_{\omega,u}$.
- Existence of the top characteristic exponent, based on Kingman's ergodic theorem 2.5.
- Establish continuity of $u \in \mathcal{U} \mapsto \boldsymbol{\mathcal{L}}_u$, and show the perturbed Taylor development (3.2.2), for $\mathcal{L}$, through ad-hoc estimates for random compositions operators on the scale $(C^s(M))_{s \leq r}$

In the rest of this section, we will consider a uniformly expanding C^r map $T : M \to M$, with dilation constant $\lambda > 1$.
Let $k \in \mathbb{N}$. Denoting by $\vec{a} := (a_1, \ldots, a_k)$ we consider the family of real cones

$$\mathcal{C}_{L,\vec{a}} := \left\{ \phi \in C^k(M),\ \phi > 0,\ \forall\ x, y \in M, \frac{\phi(x)}{\phi(y)} \leq \exp(Ld(x,y)),\ \forall\ 1 \leq j \leq k,\ \|D^j\phi(x)\| \leq a_j\phi(x) \right\} \tag{5.2.1}$$

defined for $L > 0$ and $a_1, \ldots, a_k \in \mathbb{R}_+^*$. Based on the definition, it is clear that it contains an open ball in $C^s(M)$ (around the constant function equal to 1 for example). Thus it is inner regular. The next lemma studies its outer regularity.

Lemma 5.1
For any integer k, any $\phi \in \mathcal{C}_{L,\vec{a}} \subset C^k_+(M)$, one has

$$\int_M \phi dm \geq C\|\phi\|_{C^k}$$

where the constant C depends only on $L, \vec{a}$ and $diam(M)$.

Proof. First, we start by remarking that for every $x, y \in M$, one has $e^{-Ld(x,y)} \geq e^{Ldiam(M)}$, so that for every $y \in M$, any $\phi \in \mathcal{C}_{L,\vec{a}}$,

$$\phi(y) \geq \|\phi\|_\infty e^{-Ldiam(M)} \tag{5.2.2}$$

Furthermore, for any $1 \leq j \leq k$, any $x, y \in M$, one has also

$$\|D^j\phi(x)\| \leq a_j\phi(x) \leq a_j\phi(y)e^{Ldiam(M)}$$

so that taking the supremum in $x \in M$ one obtains

$$a_j\phi(y) \geq e^{-Ldiam(M)}\|D^j\phi\|_\infty \tag{5.2.3}$$

Summing (5.2.2),...,(5.2.3) from 1 to k yields, for every $y \in M$

$$\phi(y) \geq \frac{e^{-Ldiam(M)}}{1 + a_1 + \cdots + a_k}\|\phi\|_{C^k}$$

In particular, one has

$$\int_M \phi dm \geq C \inf_{y \in M} \phi(y) \geq \frac{Ce^{-Ldiam(M)}}{1 + a_1 + \cdots + a_k}\|\phi\|_{C^k} \tag{5.2.4}$$

□

From 5.2.4 and the preliminary remark, one sees that the cones $(\mathcal{C}_{L,\vec{a}})_{L>0,\vec{a}\in\mathbb{R}_+^k}$ are **regular Birkhoff cones**. We claim that the transfer operator $\mathcal{L}_{T,g} \in L(C^r(M))$, defined by

$$\mathcal{L}_{T,g}\phi(y) = \sum_{x,Tx=y} g(x)\phi(x) \tag{5.2.5}$$

with $g \in C^r(M)$ is a **strict and uniform contraction** of the cones $\mathcal{C}_{L,\vec{a}}$, in the following sense:

Theorem 5.3 ([7] Lemma 3.2)
Let $\lambda > 1$ and $\sigma \in (\frac{1}{\lambda}, 1)$. There exists $L_0 > 0$, functions $A_j : \mathbb{R}_+^{j-1} \to \mathbb{R}_+$, $1 \le j \le k$, such that for every C^{r+1} uniformly expanding map $T : M \to M$ with dilation constant greater than λ every $L > L_0$, and $a_j \ge A_j(a_1, \dots, a_{j-1})$, one has

$$\mathcal{L}(\mathcal{C}_{L,\vec{a}}) \subset \mathcal{C}_{\sigma L,\sigma\vec{a}} \tag{5.2.6}$$

$$diam_{\mathcal{C}_{L,\vec{a}}}\mathcal{C}_{\sigma L,\sigma\vec{a}} \le 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + 2\sigma L diam(M) < +\infty \tag{5.2.7}$$

Furthermore, there exist $\rho > 0$, depending only on $L, \vec{a}, \sigma, diam(M)$ such that for every $\phi \in \mathcal{C}_{L,\vec{a}}$

$$B(\mathcal{L}_{T,g}\phi, \rho\|\mathcal{L}_{T,g}\phi\|) \subset \mathcal{C}_{L,\vec{a}} \tag{5.2.8}$$

Proof. From now on we fix $\lambda > 1$, and consider a C^r expanding map $T : M \to M$ with dilation constant greater than λ. We will note $\mathcal{L} := \mathcal{L}_{T,g}$.

For the first statement, we start by considering the case where $r = 1 + \alpha$, with $\alpha > 0$. Then one has, for every $y, y' \in M$

$$\frac{e^{g(y)}}{e^{g(y')}} = e^{g(y)-g(y')} \le e^{|g|_{C^\alpha} d(y,y')^\alpha} \tag{5.2.9}$$

We now consider $x, x' \in M$. The so-called *strong backward shadowing* property insures us that we can pair the preimages $y \in T^{-1}(x)$ and $y' \in T^{-1}(x')$ so that $d(y,y') \le \frac{1}{\lambda}d(x,x')$ (i.e the inverse branches of T are $\frac{1}{\lambda}$-contraction).
By the former remark, we have for any $\phi \in \mathcal{C}_L$

$$\mathcal{L}\phi(x) = \sum_{Ty=x} e^{g(y)}\phi(y) \le \sum_{Ty=x} e^{g(y')}\phi(y')e^{(|g|_{C^\alpha}+L)d(y,y')^\alpha} \tag{5.2.10}$$

$$\le [\sum_{Ty'=x'} e^{g(y')}\phi(y')] \exp\frac{(|g|_{C^\alpha}+L)d(x,x')^\alpha}{\lambda} = \mathcal{L}\phi(x')\exp\frac{(|g|_{C^\alpha}+L)d(x,x')^\alpha}{\lambda^\alpha} \tag{5.2.11}$$

It thus follows that for any $\sigma \in (\frac{1}{\lambda^\alpha}, 1)$, one has $\mathcal{L}(\mathcal{C}_L^*) \subset \mathcal{C}_{\sigma L}$, and that the smallest such L is given by

$$L_0 := \frac{|g|_{C^\alpha}}{\sigma - \frac{1}{\lambda^\alpha}}$$

We now consider the case of an integer $r \geq 2$, and a C^{r+1} expanding map T of M with dilation constant $\lambda > \frac{1}{\sigma}$. We want to show that $\mathcal{L}(\mathcal{C}^*_{L,\vec{a}}) \subset \mathcal{C}_{\sigma L, \sigma \vec{a}}$.
Letting $\phi \in \mathcal{C}^*_{L,\vec{a}}$ one has

$$\|D\mathcal{L}\phi(x)\| \leq \sum_{Ty=x} e^{g(y)} \frac{\|D\phi(y)\|}{\lambda} + e^{g(y)}\phi(y)\frac{\|Dg(y)\|}{\lambda} \tag{5.2.12}$$

$$\leq \sum_{Ty=x} e^{g(y)}\phi(y)\frac{a_1 + \|Dg\|_\infty}{\lambda} \tag{5.2.13}$$

$$= \mathcal{L}\phi(x)\frac{a_1 + \|Dg\|_\infty}{\lambda} \tag{5.2.14}$$

which entails that one can take

$$A_1 := \frac{\|Dg\|_\infty}{\lambda\sigma - 1}$$

For the second differential, one has

$$\|D^2\mathcal{L}\phi(x)\| \leq \sum_{Ty=x} e^{g(y)}\phi(y)\left[\frac{\|Dg(y)\|^2 + 2a_1\|Dg(y)\| + \|D^2g(y)\| + a_2}{\lambda^2} + C\frac{\|Dg(y)\| + a_1}{\lambda^3}\right] \tag{5.2.15}$$

$$\leq \mathcal{L}\phi(x)\left[\frac{\|Dg\|^2_\infty + 2a_1\|Dg\|_\infty + \|D^2g\|_\infty + a_2}{\lambda^2} + C\frac{\|Dg\|_\infty + a_1}{\lambda^3}\right] \tag{5.2.16}$$

with C a constant depending on the C^2 norm of T. It follows that one can take

$$A_2 := \frac{\|Dg\|^2_\infty + 2a_1\|Dg\|_\infty + \|D^2g\|_\infty}{\lambda^2\sigma - 1} + \frac{C}{\lambda}\frac{\|Dg\|_\infty + a_1}{\lambda^2\sigma - 1}$$

For derivative of order $j \geq 3$, one can write

$$\|D^j\mathcal{L}\phi(x)\| \leq p_j(\lambda, a_1, \dots, a_{j-1}, C, \|g\|_{C^r}) \sum_{Ty=x} e^{g(y)}\phi(y) + \frac{a_j}{\lambda^j}\sum_{Ty=x} e^{g(y)}\phi(y) \tag{5.2.17}$$

where p_j is a polynomial, which ends the proof of the first statement.

For a proof of the second statement, we refer to [7, Lemma 3.2].

The last statement follows from lemma A.3. □

Remark 5.1

- *It is noteworthy that lemma 5.3 applies to **any** C^{r+1} **expanding map** with dilation constant greater than $\lambda > 1$. In particular, if it holds for some T, it will for any $\widetilde{T} \in B_{C^{r+1}}(T, \epsilon)$ if ϵ is small enough.*

- *From lemma 5.3, we draw the (classical) conclusion that the transfer operator $\mathcal{L}_{T,g}$ of a uniformly expanding map T on M admits a **spectral gap** on $C^r(M)$, i.e there exists $\lambda_g = \rho_{sp}(\mathcal{L}_{T,g})$, $h_g \in C^r(M)$ and $\mu_g \in C^r(M)'$ such that*

$$\mathcal{L}_{T,g}h_g = \lambda_g h_g \tag{5.2.18}$$

$$\mathcal{L}^*_{T,g}\mu_g = \lambda_g\mu_g$$

$$\forall \phi \in C^r(M),\ \|\lambda_g^{-n}\mathcal{L}^n_{T,g}\phi - \langle\mu_g, \phi\rangle h_g\|_{C^r} \leq C\Delta\eta^{n-1}\|\phi\|_{C^r} \tag{5.2.19}$$

Going back to the setting of a random product of $C^{r+1}(M)$ uniformly expanding maps, $(T_{\omega,u})_{\omega\in\Omega,u\in\mathcal{U}}$, we consider the associated transfer operator cocycle above (Ω,τ), $\boldsymbol{\mathcal{L}}_u$. To insure that the results of §5.1.2 apply, and to clarify our regularity hypothesis, we will make the following assumptions on the expanding maps $(T_{\omega,u})_{\omega\in\Omega,u\in\mathcal{U}}$:

- **Uniform lower bound on the dilation constant** There exists $\lambda > 1$, such that for a.e $\omega\in\Omega$, every $u\in\mathcal{U}$, every $x\in M$ and every $v\in T_xM$

$$\|DT_{\omega,u}(x).v\| \geq \lambda\|v\| \tag{5.2.20}$$

- **Uniform boundedness of the inverse branches C^r norms** Let us recall the following standard (see [64, p.2]) fact: there is a finite family of open, connected subsets $V_1,\dots,V_p$, such that $\bigcup_{i=1}^p V_i = M$, with an associated partition of unity $(\chi_j)_{j=1\dots p}$, such that:
  - For each $1\leq j\leq p$, the map $(T_{\omega,u})_{|V_j}$ has degree $N_{\omega,j}$, with $N:\Omega\times\{1,\dots,p\}\to\mathbb{N}$ measurable.
  - For each $1\leq i\leq N_{\omega,j}$, the inverse branches of $T_{\omega,u}$ are $\frac{1}{\lambda}$ Lipschitz maps $\psi_{\omega,u,j,i}: V_j\to M$.
  - There are (random) operators $\mathcal{W}_{\omega,u,j,i}\in L(C^r(M))$, defined explicitly by

$$\mathcal{W}_{\omega,u,j,i}\phi(y) := \begin{cases}\chi_j(y)(g_{\omega,u,i}\phi)\circ\psi_{\omega,u,j,i}(y) \text{ if } y\in V_j\\ 0 \text{ otherwise}\end{cases} \tag{5.2.21}$$

    In particular, they are random composition operators, to which we may apply the results of appendix B.3.

  This allow us to formulate our assumptions: we will assume that there exists a constant $C>0$, such that for every $u\in\mathcal{U}$, every $j\in\{1,\dots,p\}$ a.e $\omega\in\Omega$, every $1\leq i\leq N_{\omega,j}$

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|\psi_{\omega,u,j,i}\|_{C^r(\mathcal{U}\times V_j)}\leq C \tag{5.2.22}$$

One may thus write the random transfer operator as a sum of the type

$$\mathcal{L}_{\omega,u}\phi = \sum_{j=1}^{p}\sum_{i=1}^{N_{\omega,j}}\mathcal{W}_{\omega,u,j,i}\phi \tag{5.2.23}$$

and extend the results of appendix B.3 to the present case. In particular, the perturbed Taylor expansion (5.1.6) is satisfied by transfer operators of random expanding maps.

5.2.1 Random physical measure and quenched linear response

In this section, we will fix the weight of the transfer operator $g := \dfrac{1}{\det(DT(.))}\in C^r(M)$. It is then a straightforward consequence of the definition of transfer operator that $\mathcal{L}_T$ preserves the integral, i.e

$$\int_M \mathcal{L}_T\phi dm = \int_M \phi dm$$

Under assumptions (5.2.20)-(5.2.22), the transfer operator cocycle associated to the random product of expanding maps takes its values in $\mathcal{M}_{\mathcal{C}_{L,\vec{a}}}(\Delta, \rho)$ for some $(L, \vec{a}, \Delta, \rho)$: this follows straightforwardly from the proof of theorem 5.3.
Assumptions (5.2.20)-(5.2.22) are satisfied for a wide range of situations, including (but not limited to) the case where one chooses at random in a i.i.d way an expanding map among a finite set $\mathcal{A}$ (i.e $\Omega = \mathcal{A}^{\mathbb{Z}}$ and τ is the full shift; this is the case e.g in theorem 1.3) or the case where one chooses at random, in a i.i.d way an expanding map in a small C^r ball $B_{C^r}(T, \epsilon)$ (i.e $\Omega = B_{C^r}(T, \epsilon)^{\mathbb{Z}}$ and τ is the associated full shift)

In this setting, the coincidence of the outer regularity form of $\mathcal{C}_{L,\vec{a}}$ and the left eigenvector of $\mathcal{L}_{\omega,u}$[3] insures us that

- For almost every $\omega \in \Omega$, every $u \in \mathcal{U}$,

$$p_{\omega,u} = \langle \ell, \mathcal{L}_{\omega,u} h_{\omega,u} \rangle = \int_M \mathcal{L}_{\omega,u} h_{\omega,u} dm = \int_M h_{\omega,u} dm = 1$$

  In particular, the fixed point $\boldsymbol{h}_u$ of $\boldsymbol{\pi}_u$ is also a fixed point for $\boldsymbol{\mathcal{L}}_u$, i.e for almost every $\omega \in \Omega$, every $u \in \mathcal{U}$,

$$\mathcal{L}_{\omega,u} h_{\omega,u} = h_{\tau\omega,u}$$

- The top characteristic exponent of this transfer operator cocycle is zero: by virtue of A.3.5, one has

$$\chi_u = \mathbb{E}[\log(\boldsymbol{p}_u)] = 0$$

Lemma 5.3, together with the results of appendix A and B, insures us that theorem 5.1 and 5.2 apply, so that one can construct a map $\boldsymbol{h} : \mathcal{U} \times \Omega \to C^r(M)$ such that:

- For every $u \in \mathcal{U}$, the map $\boldsymbol{h}_u : \Omega \to C^r(M)$ is measurable and (essentially) bounded.

- The map $u \in \mathcal{U} \mapsto \boldsymbol{h}_u \in L^\infty(\Omega, C^{r-2}(M))$ is differentiable.

The next property is a classical result, adapted to our context. A short proof is given for the reader convenience.

Proposition 5.1
The linear form

$$\nu_{\omega,u}[\phi] = \int_M \phi . h_{\omega,u} dm \tag{5.2.24}$$

defines an absolutely continuous (with respect to Lebesgue's measure) invariant probability for $T_{\omega,u}$, in the following sense: for any $\omega \in \Omega$, any $u \in \mathcal{U}$,

$$\nu_{\omega,u}[\phi \circ T_{\omega,u}] = \nu_{\tau\omega,u}[\phi] \tag{5.2.25}$$

$\boldsymbol{\nu}_u$ is called the ***random a.c.i.m*** [4].

[3] This is a very particular case, and a small miracle in itself. It is *a priori* not true in general that the outer regularity form of $\mathcal{C}_{L,\vec{a}}$ and the left eigenvector of $\mathcal{L}_{\omega,u}$ coincide.

[4] A.c.i.m stands for absolutely continuous invariant measure. Depending on the context, it is also called the random S.R.B measure (for Sinai-Ruelle-Bowen), or the random physical measure.

Proof It is the consequence of a standard computation: letting $\omega \in \Omega$, $u \in \mathcal{U}$, and $\phi \in L^1(M)$, one has

$$\nu_{\omega,u}[\phi \circ T_{\omega,u}] = \int_M \phi \circ T_{\omega,u} h_{\omega,u} dm = \int \mathcal{L}_{\omega,u}[\phi \circ T_{\omega,u} h_{\omega,u}] dm = \int \phi \mathcal{L}_{\omega,u}[h_{\omega,u}] dm \tag{5.2.26}$$
$$= \int_M \phi h_{\tau\omega,u} dm = \nu_{\tau\omega,u}[\phi]$$

since $p_{\omega,u} = \int \mathcal{L}_{\omega,u} h_{\omega,u} dm = 1$. This establishes 5.2.25. □

We have now gathered the tools to formulate the main result of this section: a **linear response** formula for the random a.c.i.m associated to a random product of expanding maps:

Theorem 5.4
Let $\boldsymbol{T} : \Omega \times \mathcal{U} \to C^r(M)$ be a (random) family of uniformly expanding maps, and let $(\boldsymbol{\mathcal{L}}_u)_{u \in \mathcal{U}}$ be the transfer operator cocycle it generates above (Ω, τ), acting on $C^{r-1}(M)$. Let $\boldsymbol{\nu}_u$ be the random measure introduced in (5.2.24).
For every observable $\phi \in L^1(M)$, almost every $\omega \in \Omega$, the map $u \in \mathcal{U} \mapsto \nu_{\omega,u}[\phi]$ is differentiable at $u = u_0$, for every $u_0 \in \mathcal{U}$ with

$$D_u \left[\int_M \phi d\nu_{\omega,u} \right]_{|u=u_0} = \sum_{n=0}^{\infty} \int_M \phi \circ T^{(n)}_{\tau^{-n}\omega,u_0} P_{\tau^{-(n+1)}\omega,u_0} h_{\tau^{-(n+1)}\omega,u_0} dm \tag{5.2.27}$$

The proof of this result will need the following lemma: one can give an estimate on the speed of convergence of the random product $\mathcal{L}^{(n)}_{\omega,u}$ towards its invariant line, analogous to the spectral gap estimate (5.2.19).

Lemma 5.2
Let $\boldsymbol{T} : \Omega \times \mathcal{U} \to C^r(M)$ be a random set of expanding maps, with dilation constants all bounded from below by some $\lambda > 1$.
Let $\boldsymbol{\mathcal{L}}_u \in L^\infty(\Omega, \mathcal{M}_{\mathcal{C}_{L,\vec{a}}}(\Delta, \rho))$ be the associated transfer operator cocycle above $(\Omega, \tau, \mathbb{P})$, and let $\boldsymbol{h}_u \in L^\infty(\Omega, C^{r-1}(M))$ its fixed point.
Then one has, for every integer $1 < s \leq r-1$, for every $\phi \in C^s(M)$,

$$\|\mathcal{L}^{(n)}_{\omega,u}\phi - h_{\tau^n\omega,u} \int_M \phi dm\|_{C^s} \leq C\eta^{n-1}\|\phi\|_{C^s} \tag{5.2.28}$$

where C depends only on the cone $\mathcal{C}_{L,\vec{a}}$ and $\eta < 1$.

Proof Let $n, p \geq 1$. We write (A.3.3), at $\phi, \mathcal{L}^{(p)}_{\tau^{-(n+p)}\omega,u}\psi$, for some $\phi, \psi \in \mathcal{C}_{L,\vec{a}} \subset C^s(M)$, such that $\int_M \psi dm = 1$, to get

$$\left\| \mathcal{L}^{(n)}_{\tau^{-n}\omega,u} \mathcal{L}^{(p)}_{\tau^{-(n+p)}\omega,u}\psi - \frac{\mathcal{L}^{(n)}_{\tau^{-n}\omega,u}\phi}{\int_M \phi dm} \right\|_{C^s} \leq \frac{\Delta}{2K}\eta^{n-1} \tag{5.2.29}$$

with K the sectional aperture of $\mathcal{C}_{L,\vec{a}}$, $\Delta = diam_{\mathcal{C}_{L,\vec{a}}}\mathcal{C}_{\sigma L,\sigma\vec{a}} < \infty$ and $\eta = \tanh\left(\frac{\Delta}{4}\right)$
Taking $p \to +\infty$, one obtains

$$\left\| h_{\omega,u} - \frac{\mathcal{L}^{(n)}_{\tau^{-n}\omega,u}\phi}{\int_M \phi dm} \right\|_{C^s} \leq \frac{\Delta}{2K}\eta^{n-1} \tag{5.2.30}$$

which yields, with the "change of variables" $\omega \leftrightarrow \tau^{-n}\omega$ and once multiplied by $0 < \int_M \phi dm \le \|\phi\|_{C^s}$, (5.2.28)

In the general case $\phi \in C^s(M)$, note that the inner regularity of the cones $\mathcal{C}_{L,\vec{a}}$ yields that $C^s(M) = \mathcal{C}_{L,\vec{a}} + (-\mathcal{C}_{L,\vec{a}})$. Furthermore, by Baire's theorem and convexity of $\mathcal{C}_{L,\vec{a}}$, it is not difficult to see that there exists a constant $0 < c < +\infty$ such that every $\phi \in C^s(M)$ decomposes into $\phi = \phi_1 - \phi_2$, with $(\phi_1, -\phi_2) \in (\mathcal{C}_{L,\vec{a}})^2$ and $\|\phi_1\|_{C^s} + \|\phi_2\|_{C^s} \le c\|\phi\|_{C^s}$. Thus,

$$\left\| h_{\tau^n\omega,u}\int_M \phi dm - \mathcal{L}^{(n)}_{\omega,u}\phi \right\|_{C^s} = \left\| h_{\tau^n\omega,u}\int_M \phi_1 dm - \mathcal{L}^{(n)}_{\omega,u}\phi_1 - \left[h_{\tau^n\omega,u}\int_M \phi_2 dm - \mathcal{L}^{(n)}_{\omega,u}\phi_2 \right] \right\|_{C^s}$$

$$\le \frac{\Delta}{2K}\eta^{n-1}\left[\|\phi_1\|_{C^s} + \|\phi_2\|_{C^s}\right] \tag{5.2.31}$$

$$\le \frac{c\Delta}{2K}\eta^{n-1}\|\phi\|_{C^s} \tag{5.2.32}$$

which conclude the proof in the general case $\phi \in C^s(M)$. □

Remark 5.2

- *From estimate (5.2.28), we draw the following conclusion: if $\phi \in C^s(M)$ is such that $\int_M \phi dm = 0$, then for any $n \ge 1$,*

$$\|\mathcal{L}^{(n)}_{\tau^{-n}\omega,u}\phi\|_{C^s} \le \frac{c\Delta}{2K}\eta^{n-1}\|\phi\|_{C^s}$$

  In particular, the limit $\sum_{n=0}^{\infty}\mathcal{L}^{(n)}_{\tau^{-n}\omega,u}\phi$ is well defined whenever $\int_M \phi dm = 0$.

- *Estimate (5.2.28) has far reaching consequences: in particular, it can be used to establish* exponential decay of random correlations, *in the same way (5.2.19) yields exponential decay of correlations in the deterministic case (see (2.4.2)). We refer to [7, Thm B] for more details on this.*

Proof of theorem 5.4 To show the regularity of $u \in \mathcal{U} \mapsto \nu_{\omega,u}[\phi]$, our strategy is to use a "regularity under the integral" type of result. It follows from (5.1.7) and (5.3.13) that the map $u \in \mathcal{U} \mapsto \boldsymbol{h}_u \in L^\infty(\Omega, C^{r-2}(M))$ is differentiable with

$$D_u\boldsymbol{h}_u = \left[\mathbb{1} - \boldsymbol{Q}^{(0,1)}_{\boldsymbol{\pi}}(u, \boldsymbol{h}_u)\right]^{-1}\boldsymbol{Q}^{(1,0)}_{\boldsymbol{\pi}}(u, \boldsymbol{h}_u) \tag{5.2.33}$$

In our particular case, one may introduce $\boldsymbol{Q}_u := \boldsymbol{Q}^{(0,1)}_{\boldsymbol{\pi}}(u, \boldsymbol{h}_u)$, such that

$$(\boldsymbol{Q}_u\cdot)_\omega = \mathcal{L}_{\tau^{-1}\omega,u} - \left(\int_M .dm\right) h_{\omega,u} \tag{5.2.34}$$

$$Q^{(1,0)}_{\pi}(\omega, u, h_{\omega,u}) = P_{\tau^{-1}\omega,u}h_{\tau^{-1}\omega,u} - \left(\int_M P_{\tau^{-1}\omega,u}h_{\tau^{-1}\omega,u}dm\right) h_{\omega,u} \tag{5.2.35}$$

The normalization $\int_M \boldsymbol{h}_u dm = 1$ implies that $\int_M D_u\boldsymbol{h}_u dm = 0$. Similarly, $\int_M \boldsymbol{\mathcal{L}}_u\boldsymbol{h}_u dm = 1$ yields that

$$0 = D_u[\int_M \mathcal{L}_{\omega,u}h_{\omega,u}] = \int_M P_{\omega,u}h_{\omega,u}dm + \int_M \mathcal{L}_{\omega,u}D_uh_{\omega,u}dm = \int_M P_{\omega,u}h_{\omega,u}dm$$

Hence, one obtains $Q_{\omega,u}D_u h_{\omega,u} = \mathcal{L}_{\tau^{-1}\omega,u}h_{\tau^{-1}\omega,u}$ and $Q_\pi^{(1,0)}(\omega,u,h_{\omega,u}) = P_{\tau^{-1}\omega,u}h_{\tau^{-1}\omega,u}$

A simple computation shows that

$$(\boldsymbol{Q}_u^n)_\omega = Q^{(n)}_{\tau^{-n}\omega,u} = \mathcal{L}^{(n)}_{\tau^{-n}\omega,u} - \left(\int_M .dm\right) h_{\omega,u}$$

and thus by (5.2.28), for any (essentially) bounded $\phi : \Omega \to C^{r-1}(M)$, one has

$$\left([\mathbb{1} - \boldsymbol{Q}_u]^{-1}\phi\right)_\omega = \sum_{n=0}^{\infty} Q^{(n)}_{\tau^{-n}\omega,u}\phi_{\tau^{-n}\omega} \tag{5.2.36}$$

Let $u_0 \in \mathcal{U}$. From this last equation and remark 5.2, one sees that (5.2.33) can be rewritten as

$$D_u h_{\omega,u_0} = \sum_{n=0}^{\infty} \mathcal{L}^{(n)}_{\tau^{-n}\omega,u_0} P_{\tau^{-(n+1)}\omega,u_0} h_{\tau^{-(n+1)}\omega,u_0} \tag{5.2.37}$$

which is valid both in L^∞ and for almost every $\omega \in \Omega$. This last equation yields

$$D_u \nu_{\omega,u_0}[\phi] = \int_M \phi D_u h_{\omega,u_0} dm = \int_M \sum_{n=0}^{\infty} \phi \mathcal{L}^{(n)}_{\tau^{-n}\omega,u_0} P_{\tau^{-(n+1)}\omega,u_0} h_{\tau^{-(n+1)}\omega,u_0} dm \tag{5.2.38}$$

which yields (5.2.27) by using boundedness of the integral on $C^s(M)$ and the duality property of the transfer operator $\int_M \phi \mathcal{L}_{\omega,u}\psi dm = \int_M \phi \circ T_{\omega,u}\psi dm$. $\square$

We can also give an annealed version of theorem 5.4:

Theorem 5.5
Let $r \geq 5$, $\boldsymbol{T} : \Omega \times \mathcal{U} \to C^r(M)$ be a (random) family of uniformly expanding maps, and let $(\boldsymbol{\mathcal{L}}_u)_{u\in\mathcal{U}}$ be the transfer operator cocycle it generates above (Ω, τ), acting on $C^{r-1}(M)$. Let $\boldsymbol{\nu}_u$ be the random measure introduced in (5.2.24).
For every observable $\phi \in L^1(M)$, the map $u \in \mathcal{U} \mapsto \mathbb{E}\left[\int_M \phi d\boldsymbol{\nu}_u\right]$ is differentiable at $u = u_0$, for every $u_0 \in \mathcal{U}$ with

$$D_u \left[\mathbb{E}\left(\int_M \phi d\boldsymbol{\nu}_u\right)\right]_{u=u_0} = \sum_{n=0}^{\infty} \int_\Omega \int_M \phi \circ T^{(n)}_{\omega,u_0} P_{\tau^{-1}\omega,u_0} h_{\tau^{-1}\omega,u_0} dm d\mathbb{P} \tag{5.2.39}$$

Proof of theorem 5.5 The proof builds on theorem 5.4 and differentiation under the integral. It follows from theorems 5.1 and 5.2 that if one sees $h_u \in L^\infty(\Omega, C^{r-1}(M))$ as an application $h_u \in L^\infty(\Omega, C^{r-4}(M))$ is twice differentiable with respect to $u \in \mathcal{U}$.
Let $u_0 \in \mathcal{U}$. One can apply "differentiation under the integral" for the quantity $\mathbb{E}\left[\int_M \phi d\boldsymbol{\nu}_u\right]$, since

- $\int_M \phi h_{\tau^{-1}\omega,u} dm$ is differentiable (interpreting $h_{\tau^{-1}\omega,u}$ as an element of $C^{r-3}(M)$)
- $D_u\left[\int_M \phi h_{\tau^{-1}\omega,u} dm\right]_{u=u_0}$ is dominated by some element of $L^1(\Omega)$, since $u \in \mathcal{U} \mapsto D_u h_{\omega,u}$ is continuous, and thus bounded, in a neighborhood of u_0 (interpreting $h_{\tau^{-1}\omega,u}$ as an element of $C^{r-4}(M)$)

Thus, one gets that $u \in \mathcal{U} \mapsto \mathbb{E}\left[\int_M \phi d\boldsymbol{\nu}_u\right]$ is differentiable at $u = u_0$, with

$$\begin{aligned}
D_u\left[\mathbb{E}\left(\int_M \phi d\boldsymbol{\nu}_u\right)\right]_{u=u_0} &= \int_\Omega \int_M \phi D_u h_{\omega,u_0} dm d\mathbb{P} \\
&= \int_\Omega \sum_{n=0}^{\infty} \int_M \phi \mathcal{L}^{(n)}_{\tau^{-n}\omega,u_0} P_{\tau^{-n-1}\omega,u_0} h_{\tau^{-n-1}\omega,u_0} dm d\mathbb{P} \\
&= \sum_{n=0}^{\infty} \int_\Omega \int_M \phi \circ T^{(n)}_{\tau^{-n}\omega,u_0} P_{\tau^{-n-1}\omega,u_0} h_{\tau^{-n-1}\omega,u_0} dm d\mathbb{P} \\
&= \sum_{n=0}^{\infty} \int_\Omega \int_M \phi \circ T^{(n)}_{\omega,u_0} P_{\tau^{-1}\omega,u_0} h_{\tau^{-1}\omega,u_0} dm d\mathbb{P}
\end{aligned} \tag{5.2.39}$$

where we used the change of variables $\omega = \tau^{-n}\omega$. Exchanging the infinite sum and $\mathbb{E}$ is justified by boundedness of the latter linear form on $L^\infty(\Omega, C^0(M))$. □

Remark 5.3
The lower bound $r \geq 5$ on the regularity in theorem 5.5 is an artifact of the proof, and certainly not optimal: we expect the result to hold for $r > 4$.

5.2.2 Mean and variance in the central limit theorem

In this section, we wish to study regularity, with respect to parameters, of the mean and variance of the (quenched) central limit theorem satisfied by a random product of expanding maps (see e.g [18, Theorem B])

Our strategy is to use tools from analytical perturbation theory by introducing (yet another) small complex parameter in the weight of the transfer operator $\mathcal{L}_{T_{\omega,u}, g_{\omega,u}}$, with $g_{\omega,u} := -\log(|\det(DT_{\omega,u})|)$.
More precisely, given $A \in L^\infty(\Omega, C^r(M))$, and $t \in \mathbb{D}(0,\epsilon) \subset \mathbb{C}$, we define

$$\mathcal{L}_{\omega,t,u}\phi(x) := \mathcal{L}_{\omega,u}\left(e^{tA}\phi\right) = \sum_{y \in T^{-1}_{\omega,u}x} e^{g_{\omega,u}y + tA_\omega y}\phi(y) \tag{5.2.40}$$

It is easy to see that the map $t \in \mathbb{D}(0,\epsilon) \mapsto \boldsymbol{\mathcal{L}}_{t,u} \in L^\infty(\Omega, C^r(M))$ is analytical.
The major issue with this approach is that it forces us to work with complex-valued functions and operators: in such a setting, the results of classical cone contraction theory (that is, Birkhoff approach) no longer apply.
Fortunately, an analogous theory for "complex cones" was developed by Rugh and Dubois (see [71]), which extends cone contraction results (notably spectral gap existence and estimates) to complex Banach spaces. We recall the basic theory of $\mathbb{C}$-cones in appendix A.

Here, we will be working with the so-called **canonical complexification** of the previously introduced family of cones $(\mathcal{C}_{L,\vec{a}})_{L>0,\vec{a}\in\mathbb{R}^r_+}$. We recall basic facts on the canonical complexification of a (real) Banach space and a Birkhoff cone in §A.2.1 (see also the seminal papers [21, 71]).

In particular, it follows from theorem A.7 that if $\mathcal{L}$ is a strict and uniform contraction of the Birkhoff cone $\mathcal{C}$, then its canonical complexification is also a strict and uniform contraction

of $\mathcal{C}_\mathbb{C}$. This fact is instrumental in the proof of theorem 5.5, as it has the following (obvious) consequence: any *analytical* perturbation $t \in \mathbb{D}(0,\epsilon) \mapsto \mathcal{L}_t$ of the original operator $\mathcal{L}$ is also a strict and uniform contraction of $\mathcal{C}_\mathbb{C}$, as long as one choose ϵ sufficiently small.
In our case we will take ϵ so small that for any $t \in \mathbb{D}(0,\epsilon)$

$$\|\mathbb{1} - e^{tA}\|_{L^\infty(\Omega, L(C^r(M)))} < \frac{\rho}{4} \tag{5.2.41}$$

where ρ is given by lemma A.3.

Going back to our random product of expanding maps, and to the perturbed transfer operator (5.2.40), we consider its characteristic exponent, given as usual by (A.3.5):

$$\chi_{t,u} = \mathbb{E}\left[\log(|\boldsymbol{p}_{t,u}|)\right] \tag{5.2.42}$$

The next theorem studies its regularity at $t=0$.

Lemma 5.3
Consider the cocycle above (Ω,τ) generated by the perturbed transfer operator (5.2.40). Let $\chi_{t,u} \in \mathbb{R}$ be its characteristic exponent. Then the map $t \in \mathbb{D}(0,\epsilon) \mapsto \chi_{t,u}$ is analytical, and its first and second derivative at $t=0$ are given by:

$$\left[\frac{d\chi_{t,u}}{dt}\right]_{t=0} = \mathbb{E}\left[\int_M A h_u dm\right] \tag{5.2.43}$$

$$\left[\frac{d^2\chi_{t,u}}{dt^2}\right]_{t=0} = \int_\Omega \int_M A_\omega^2 h_{\omega,u} dm d\mathbb{P} + 2\sum_{n=1}^{\infty} \int_\Omega \int_M A_{\tau^n\omega} \circ T^{(n)}_{\omega,u}.A_\omega h_{\omega,u} dm d\mathbb{P} - \int_\Omega \left(\int_M A_\omega h_{\omega,u_0} dm\right)^2 d\mathbb{P} \tag{5.2.44}$$

In particular, the right-hand terms above are twice differentiable with respect to $u \in \mathcal{U}$.

Proof of lemma 5.3 It follows from theorem A.7 and our requirement on the size of the perturbation (5.2.41) that one can apply [71, Theorem 10.2] to this situation: for any fixed $u \in \mathcal{U}$, the map $t \in \mathbb{D}(0,\epsilon) \mapsto \chi_{t,u}$ is analytic.

Furthermore, going over the proof of theorem 10.2 in [71], one sees that this follows from analyticity of the maps $t \mapsto \boldsymbol{p}_{t,u} := \int_M \boldsymbol{\mathcal{L}}_{t,u}\boldsymbol{h}_{t,u} dm$, and $t \mapsto \boldsymbol{h}_{t,u} \in L^\infty(\Omega, C^{r-1}(M))$ where $\boldsymbol{h}_{t,u} \in L^\infty(\Omega, C^r(M))$ is the generalized eigenvector of $\boldsymbol{\mathcal{L}}_{t,u}$ (i.e the fixed point of $\boldsymbol{\pi}_{t,u}$), normalized by $\int_M h_{\omega,t,u} dm = 1$.
One can use (A.3.5) and take the derivative, to obtain:

$$\frac{d\chi_{t,u}}{dt} = \mathbb{E}\left[\frac{1}{\boldsymbol{p}_{t,u}}\frac{d\boldsymbol{p}_{t,u}}{dt}\right] \tag{5.2.45}$$

Now, one has

$$\begin{aligned}
\frac{dp_{\tau\omega,t,u}}{dt} &= \frac{d}{dt}\left[\int_M \mathcal{L}_{\omega,t,u} h_{\omega,t,u} dm\right] \\
&= \int_M \frac{d\mathcal{L}_{\omega,t,u}}{dt} h_{\omega,t,u} dm + \int_M \mathcal{L}_{\omega,t,u}\frac{dh_{\omega,t,u}}{dt} dm \\
&= \int_M \mathcal{L}_{\omega,t,u}\left[A_\omega h_{\omega,t,u} + \frac{dh_{\omega,t,u}}{dt}\right] dm
\end{aligned}$$

Take $t=0$: using the duality property of $\mathcal{L}_{\omega,u}$ and the normalization $\int_M h_{\omega,t,u}dm = 1$, one obtains

$$\left[\frac{dp_{\tau\omega,t,u}}{dt}\right]_{t=0} = \int_M A_\omega h_{\omega,u} dm \tag{5.2.46}$$

Injecting this last equality in (5.2.45) (together with $p_{\omega,u}=1$ for all $(\omega,u)\in\Omega\times\mathcal{U}$ and the τ-invariance of $\mathbb{P}$), we get

$$\left[\frac{d\chi_{t,u}}{dt}\right]_{t=0} = \int_\Omega\int_M A_\omega h_{\omega,u} dm d\mathbb{P} \tag{5.2.43}$$

Note that one can use this to recover the regularity of $u\in\mathcal{U}\mapsto\mathbb{E}\left[\int_M\phi d\boldsymbol{\nu}_u\right]$. However, it should be noted that it is *not enough* to recover the linear response formula (5.2.39): one still needs to justify the differentiation under the integral.

For the second derivative, we start by introducing

$$\Sigma_u^2 := \int_\Omega\int_M A_\omega^2 h_{\omega,u} dm d\mathbb{P} + 2\sum_{n=1}^{\infty}\int_\Omega\int_M A_{\tau^n\omega}\circ T_{\omega,u}^{(n)}.A_\omega h_{\omega,u} dm d\mathbb{P}$$

We now have

$$\frac{d^2\chi_{t,u_0}}{dt^2} = \mathbb{E}\left[-\left(\frac{\boldsymbol{p}'_{t,u_0}}{\boldsymbol{p}_{t,u_0}}\right)^2 + \frac{\boldsymbol{p}''_{t,u_0}}{\boldsymbol{p}_{t,u_0}}\right] \tag{5.2.47}$$

From (5.2.45), we already know that $\frac{dp_{\tau\omega,t,u_0}}{dt} = \int_M \mathcal{L}_{\omega,t,u_0}\left[A_\omega h_{\omega,t,u_0}\right]dm + \int_M \mathcal{L}_{\omega,t,u_0}\partial_t h_{\omega,t,u_0} dm$. Taking another derivative with respect to t, one gets (letting the u_0 indices out to keep the notations readable)

$$\frac{d^2p_{\tau\omega,t}}{dt^2} = \int_M \partial_t\mathcal{L}_{\omega,t}\left[A_\omega h_{\omega,t} + A_\omega\partial_t h_{\omega,t}\right]dm + \int_M \partial_t\mathcal{L}_{\omega,t}(\partial_t h_{\omega,t})dm + \int_M \mathcal{L}_{\omega,t}\partial_t^2 h_{\omega,t} dm$$

For $t=0$, one obtains (noting that $\partial_t\mathcal{L}_{\omega,0}(.) = \mathcal{L}_{\omega,u}(A_\omega .)$, and that $\int_M\partial_t^2 h_{\omega,t}dm = 0$)

$$\left[\frac{d^2p_{\tau\omega,t}}{dt^2}\right]_{t=0} = \int_M A_\omega^2 h_{\omega,u_0} dm + 2\int_M A_\omega[\partial_t h_{\omega,t}]_{t=0} dm \tag{5.2.48}$$

In the second term of the right-hand term, we can give an explicit expression for $[\partial_t h_{\omega,t}]_{t=0}$, using formula (5.2.37):

$$[\partial_t h_{\omega,t}]_{t=0} = \sum_{n=0}^{\infty}\mathcal{L}_{\tau^{-n}\omega,u_0}^{(n)}\mathcal{L}_{\tau^{-n-1}\omega,u_0}\left[A_{\tau^{-n-1}\omega}h_{\tau^{-n-1}\omega,u_0}\right] \tag{5.2.49}$$

$$= \sum_{n=1}^{\infty}\mathcal{L}_{\tau^{-n}\omega,u_0}^{(n)}\left[A_{\tau^{-n}\omega}h_{\tau^{-n}\omega,u_0}\right] \tag{5.2.50}$$

It thus follows that

$$\left[\frac{d^2p_{\tau\omega,t}}{dt^2}\right]_{t=0} = \int_M A_\omega^2 h_{\omega,u_0} dm + 2\sum_{n=1}^{\infty}\int_M A_\omega\circ T_{\tau^{-n}\omega,u_0}^{(n)} A_{\tau^{-n}\omega}h_{\tau^{-n}\omega,u_0} dm \tag{5.2.51}$$

Re-injecting in (5.2.47), one gets

$$\left[\frac{d^2\chi_{t,u_0}}{dt^2}\right]_{t=0} = \Sigma^2_{u_0} - \int_\Omega \left(\int_M A_\omega h_{\omega,u_0} dm\right)^2 d\mathbb{P} \tag{5.2.52}$$

From theorem 5.2, we deduce that there exists $\epsilon, \delta > 0$ (both independent of u) such that the map $(t,u) \in \mathbb{D}(0,\epsilon) \times B(u_0,\delta) \mapsto \chi_{t,u}$ is twice differentiable; furthermore, analytic perturbation theory (here, [71, Theorem 10.2]) gives us a $\epsilon' > 0$, such that for every $u \in B(u_0,\delta)$, $t \in \mathbb{D}(0,\epsilon') \mapsto \chi_{t,u}$ is analytic, so that one can write Cauchy's formulas:

$$\left[\frac{d\chi_{t,u}}{dt}\right]_{t=0} = \int_{C(0,\epsilon')} \frac{\chi(s,u)}{s^2} ds \tag{5.2.53}$$

$$\left[\frac{d^2\chi_{t,u}}{dt^2}\right]_{t=0} = \int_{C(0,\epsilon')} \frac{\chi(s,u)}{s^3} ds \tag{5.2.54}$$

The result now follow from differentiation under the integral. □

5.2.3 Application: Hausdorff dimension of repellers for 1D expanding maps

In this section we are interested in the random product of one-dimensional maps, with uniform dilation but not necessarily defined everywhere. More precisely, we are interested in the following class of systems:

Definition 5.2
*Let $I_1, \dots I_N \subset [0,1]$ be disjoints intervals, and $r \geq 2$. A C^r map $T : I_1 \cup \dots \cup I_N \to [0,1]$ is called a **cookie-cutter** if it satisfies the following conditions:*

- *There exists some $\lambda > 1$ such that $\inf |T'| \geq \lambda$*
- *For each $i \in \{1, \dots, n\}$, $T(I_i) = [0,1]$*

*If T is a cookie-cutter, we introduce its **repeller**,*

$$\Lambda := \{x \in I_1 \cup \dots \cup I_N, T^n(x) \text{ is well-defined for all } n\} = \bigcap_{i=1}^{\infty} T^{-i}([0,1]) \tag{5.2.55}$$

We will denote by $CC^r([0,1])$ the set of all C^r cookie-cutters.

In other words, a cookie-cutter is a one-dimensional expanding map with full branches. It is a well-known fact that the repeller associated to such a map is a Cantor set. How can one perturb such a map ? It is a priori not clear as perturbing the map might change the intervals of definition $I_1, \dots, I_N$. To circumvent that difficulty, we consider perturbations of the inverse branches of a cookie-cutter.

Definition 5.3
Let $\mathcal{U}$ be an open subset of some Banach space $\mathcal{B}$, and let $\psi_i : \mathcal{U} \times [0,1] \to (0,1)$, $i \in \{1, \dots, N\}$ be C^r maps such that

- *For every $i \in \{1, \dots, N\}$, $\|\partial_x \psi_i\|_\infty \leq 1/\lambda < 1$*

- *For every $i \in \{1, \ldots, N\}$, every $u \in \mathcal{U}$, the intervals $I_{i,u} = \psi_{i,u}([0,1])$ are pairwise disjoints.*

This data defines a cookie-cutter map T_u on $I_{1,u}, \ldots, I_{N,u}$, by $T_u = \psi_{i,u}^{-1}$ on $I_{i,u}$. We call it a perturbed cookie-cutter.

The question we want to study is the following: if one were to choose at each step a random cookie-cutters, and then perturb it in the sense of definition 5.3, does the Hausdorff dimension of the repeller change in a smooth way ?

The tool we propose to use to answer that question is a random version of the celebrated *Bowen formula*, which connects the transfer operator cocycle's top characteristic exponent and the Hausdorff dimension of the associated (random) repeller (cf. theorem 5.6).

More precisely, one considers a random product $T_{\omega,u}^{(n)} := T_{\tau^{n-1}\omega,u} \circ \cdots \circ T_{\omega,u}$, with $\boldsymbol{T} : \mathcal{U} \to L^\infty(\Omega, CC^r([0,1]))$. We assume furthermore that $\sup_{u \in \mathcal{U}} \mathbb{E}\left[\log \|\boldsymbol{T}'_u\|\right] < +\infty$, and that assumptions (5.2.20) and (5.2.22) are satisfied.

Associated to this random product is a random repeller, defined by

$$\Lambda_{\omega,u} := \bigcap_{i=1}^{\infty} \left(T_{\omega,u}^{(i)}\right)^{-1}([0,1]) \tag{5.2.56}$$

Given $s \geq 0$, we also define the transfer operator $\mathcal{L}_{\omega,s,u}$ by

$$\mathcal{L}_{\omega,s,u}\phi(x) := \sum_{T_{\omega,u}y = x} \frac{1}{|T'_{\omega,u}(y)|^s}\phi(y) \tag{5.2.57}$$

It follows from theorem 5.3 that $\boldsymbol{\mathcal{L}}_{s,u}$ takes its values in $\mathcal{M}_{\mathcal{C}_{L,\bar{a}}}(\Delta, \rho)$ for some $\Delta < +\infty$ and $\rho > 0$. It is also clear from the definition that $\mathcal{L}_{\omega,s,u}$ depends analytically of $s \geq 0$ (up to considering a small complex extension of s), so that it follows from [71, Theorem 10.2] that for every fixed $u_0 \in \mathcal{U}$, the map $s \mapsto \chi_{s,u_0}$ is analytic.

We also introduce the following quantities:

$$M_n(\omega, s, u) := \sup_{y \in \Lambda_{n,\omega,u}} \mathcal{L}_{\omega,s,u}^{(n)}\mathbf{1}(y) \tag{5.2.58}$$

$$m_n(\omega, s, u) := \inf_{y \in \Lambda_{n,\omega,u}} \mathcal{L}_{\omega,s,u}^{(n)}\mathbf{1}(y) \tag{5.2.59}$$

where $\Lambda_{\omega,n,u} := \bigcap_{i=n}^{\infty} \left(T_{\omega,u}^{(i)}\right)^{-1}([0,1])$, and finally we let

$$-\infty \leq \underline{P}(\omega, s, u) := \liminf \frac{1}{n}\log(m_n(\omega, s, u)) \leq \overline{P}(\omega, s, u) := \limsup \frac{1}{n}\log(M_n(\omega, s, u)) \leq +\infty$$

Those last quantities exists by super-multiplicativity (resp. sub-multiplicativity) and Kingman's ergodic theorem 2.5, and are $\mathbb{P}$-almost surely constant by ergodicity of τ.
One can show that those quantities almost surely agree, their common value being $\chi_{s,u}$ the top characteristic exponent of the random product, and that $s \geq 0 \mapsto \chi_{s,u} + s\log(\lambda) \in \mathbb{R}$ is strictly decreasing (see [70, Lemma 3.5 and Theorem 4.4]).
Furthermore, this strictly decreasing map admits a unique zero that coincide with the (a.s) Hausdorff dimension of the random repeller $\Lambda_{\omega,u}$ (see [70, Theorem 4.4 and 5.3]):

Theorem 5.6 ([72] Theorem 5.3)
Let τ be an invertible and ergodic map of $(\Omega, \mathbb{P})$. Let $\boldsymbol{T} \in L^\infty(\Omega, CC^r([0,1]))$ be a random product of cookie-cutters, such that $\mathbb{E}\left[\log \|T'_\omega\|_\infty\right] < \infty$.
Then $\mathbb{P}$-almost surely the Hausdorff dimension of the random repeller Λ_ω is given by the unique zero $z(\boldsymbol{T})$ of the top characteristic exponent χ_s of the transfer operator cocycle $\boldsymbol{\mathcal{L}}_s$.

For a proof, we refer to [72, §4-5]. The question is now the dependence of that zero on the parameter u :

Theorem 5.7
Let $\boldsymbol{T} \in \Omega \times \mathcal{U} \to CC^r([0,1])$ be a random cookie-cutter, such that for a.e $\omega \in \Omega$, the map $u \in \mathcal{U} \mapsto T_{\omega,u} \in C^r([0,1])$ is C^s for some $s > 1$. Assume furthermore that (5.2.20) and(5.2.22) are satisfied.
Then the Hausdorff dimension of the random repeller defined by (5.2.56) is C^s with respect to $u \in \mathcal{U}$.

Proof of theorem 5.7 Theorem 5.6 entails that the Hausdorff dimension of $\Lambda_{\omega,u}$ is given by some $z(u)$ such that $\chi_{z(u),u} = 0$. The most natural tool to investigate the question of the parameter dependency of z is the implicit function theorem:
From theorem 5.6 and 5.2, one has that

- For every $u \in \mathcal{U}$, $\chi_{z(u),u} = 0$
- The map $(s,u) \mapsto \chi_{s,u}$ is C^{r-2}.

The only assumption left in the implicit function theorem is that $\partial_s \chi$ at (s, u_0) is non-zero, which follows straightforwardly from the fact that for fixed $u \in \mathcal{U}$, $s \geq 0 \mapsto \chi_{s,u} + s\log(\lambda) \in \mathbb{R}$ is strictly decreasing. $\square$

5.3 Proof of the main results

5.3.1 Lipschitz regularity of the top characteristic exponent: a proof of theorem 5.1

In this section, we want to establish quantitative results concerning continuity of the map $u \in \mathcal{U} \mapsto \chi_u$. In order to do that, we will use the "convenient representation" (A.3.5) of the characteristic exponent exploiting the fixed point $\boldsymbol{h}_u$ and the regularity property of the cone $\mathcal{C}_s \subset \mathcal{B}_s,\ s = 0, 1$.

Let $u_0 \in \mathcal{U}$, $s \in \{0,1\}$. By assumption (ii) in theorem 5.1, there exists some regulars Birkhoff cones $\mathcal{C}_s \subset \mathcal{B}_s$, such that for $u \in \mathcal{U}$ close enough to u_0, the operators $\mathcal{L}_{\omega,u}$ are strict and uniform contractions of the cone $\mathcal{C}_s$, i.e there are Δ independent of $\omega \in \Omega$, $\rho > 0$ such that

$$\begin{cases} \mathcal{L}_{\omega,u}(\mathcal{C}_s^*) \subsetneq \mathcal{C}_s^* \\ diam_{\mathcal{C}_s}(\mathcal{L}_{\omega,u}(\mathcal{C}_s^*)) \leq \Delta \\ B_s(\mathcal{L}_{\omega,u}\phi, \rho\|\mathcal{L}_{\omega,u}\phi\|_s) \subset \mathcal{C}_s \end{cases}$$

As such, one can readily apply the results of appendix A, and consider the fixed point $\boldsymbol{h}_u \in L^\infty(\Omega, \mathcal{B}_s)$ of the projection $\boldsymbol{\pi}_u$ defined by (5.1.5).

Let $u, v \in \mathcal{U}'$ where $\mathcal{U}' \subset \mathcal{U}$ is a (small enough) neighborhood of u_0. Then one can write, for any $n \in \mathbb{N}$, $\omega \in \Omega$

$$h_{\tau^n\omega,u} - h_{\tau^n\omega,v} = \pi^{(n)}_{\omega,u}(h_{\omega,u}) - \pi^{(n)}_{\omega,v}(h_{\omega,v}) \tag{5.3.1}$$

Taking the $\mathcal{B}_0$ norm, and using a standard trick, one obtains:

$$\begin{aligned} \|h_{\tau^n\omega,u} - h_{\tau^n\omega,v}\|_{\mathcal{B}_0} &\leq \|\pi^{(n)}_{\omega,u}(h_{\omega,u}) - \pi^{(n)}_{\omega,v}(h_{\omega,u})\|_{\mathcal{B}_0} + \|\pi^{(n)}_{\omega,v}(h_{\omega,u}) - \pi^{(n)}_{\omega,v}(h_{\omega,v})\|_{\mathcal{B}_0} && (5.3.2) \\ &\leq \|\pi^{(n)}_{\omega,u}(h_{\omega,u}) - \pi^{(n)}_{\omega,v}(h_{\omega,u})\|_{\mathcal{B}_0} + K\eta^{n-1}\Delta && (5.3.3) \end{aligned}$$

with the second inequality coming from the continuous inclusion $\mathcal{B}_1 \hookrightarrow \mathcal{B}_0$ and (A.3.3). For the other term, we use assumption (iii) on the Lipschitz continuity of $u \in \mathcal{U} \mapsto \boldsymbol{\mathcal{L}}_u \in L^\infty(\Omega, L(\mathcal{B}_1, \mathcal{B}_0))$: by a straightforward induction it extends to $u \in \mathcal{U} \mapsto \boldsymbol{\pi}^n_u$, so that one gets

$$\|\pi^{(n)}_{\omega,u}(h_{\omega,u}) - \pi^{(n)}_{\omega,v}(h_{\omega,u})\|_{\mathcal{B}_0} \leq C\|u - v\|\|\boldsymbol{h}_u\|_{L^\infty(\Omega,\mathcal{B}_1)} \tag{5.3.4}$$

with $C = C(u_0)$ independent of $\omega \in \Omega$. Therefore, for n big enough we have that

$$\|\boldsymbol{h}_u - \boldsymbol{h}_v\|_{L^\infty(\Omega,\mathcal{B}_0)} \leq C\|u - v\|\|\boldsymbol{h}_u\|_{L^\infty(\Omega,\mathcal{B}_1)} + \epsilon \tag{5.3.5}$$

and it follows that $u \in \mathcal{U}' \mapsto \boldsymbol{h}_u \in L^\infty(\Omega, \mathcal{B}_0)$ is locally Lipschitz, as announced.

Now consider, for $s \in \{0, 1\}$, the normalization factor $p_{s,\omega,u} = \langle \ell_s, \mathcal{L}_{\omega,u} h_{\omega,u} \rangle$. By assumption (iv), one has $p_{1,\omega,u} = p_{0,\omega,u} = p_{\omega,u}$ is independent of the chosen norm; thus the top characteristic exponent χ_u is too by (A.3.5).

By the regularity property of $\mathcal{C}_s$, this quantity stays strictly positive for u close enough to u_0, so that its logarithm is well-defined. Furthermore, by Lipschitz continuity of $u \in \mathcal{U} \mapsto \boldsymbol{\mathcal{L}}_u \boldsymbol{h}_u \in \mathcal{L}^\infty(\Omega, B_0)$, one obtains Lipschitz continuity of $u \in \mathcal{U} \mapsto \log(\boldsymbol{p}_u)$.
It follows immediately from (A.3.5) that

$$\chi_u - \chi_v = \int_\Omega \log\left(\frac{\boldsymbol{p}_u}{\boldsymbol{p}_v}\right) d\mathbb{P} \tag{5.3.6}$$

and thus (by dominated convergence) $u \in \mathcal{U} \mapsto \chi_u$ is Lipschitz continuous. □

5.3.2 Differentiability of the top characteristic exponent: a proof of theorem 5.2

We now turn to theorem 5.2. We want to establish the differentiability of $u \in \mathcal{U} \mapsto \chi_u$, and for that we will use the strategy outlined in § 5.1.2, i.e studying the regularity of $u \in \mathcal{U} \mapsto \boldsymbol{h}_u \in L^\infty(\Omega, \mathcal{B}_{s-1})$ with theorem 3.1.

The only point left to prove is the invertibility (and boundedness of the inverse) of the partial derivative $\boldsymbol{Q}_0$ of $\boldsymbol{\pi}_u$ at $u_0 \in \mathcal{U}$, on $L^\infty(\Omega, \mathcal{B}_{s-1})$ (the first point was to establish continuity of $u \in \mathcal{U} \mapsto \boldsymbol{h}_u \in L^\infty(\Omega, \mathcal{B}_s)$ for some $1 \leq s < r$, which was done in theorem 5.1; the second follows straightforwardly from assumption (5.1.6)).

Recall that for any $s \in (0, r]$, the operators $\mathcal{L}_{\omega,u}$ contracts almost surely a regular $\mathbb{R}$-cone $\mathcal{C}_{s,u_0}$ for $u \in \mathcal{U}$ close enough to u_0. We also recall that $\mathcal{C}_{s,u_0} = j_{s,t}(\mathcal{C}_{t,u_0})$ for any $0 < s \leq t$, and that the linear forms $\ell_{s,0}$ given by the regularity property of $\mathcal{C}_{s,u_0}$ boundedly extend one another.

Note that if $y = x + tv$, $v \in \mathcal{B}_s$, $s > 1$, for $|t| < r$ lemma A.1 gives us

$$d_{\mathcal{C}_s}(x, x+tv) \leq \frac{2\|tv\|_s}{r} + o(|t|) \tag{5.3.7}$$

Using lemma A.2, one obtains the following estimate in $\mathcal{B}_{s-1}$ norm, for a.e $\omega \in \Omega$, for any $\phi, z \in L^\infty(\Omega, \mathcal{B}_s)$,

$$\|\pi^{(n)}_{\tau^{-n}\omega,u}(\phi_{\tau^{-n}\omega}) - \pi^{(n)}_{\tau^{-n}\omega,u}(\phi_{\tau^{-n}\omega} + tz_{\tau^{-n}\omega})\|_{s-1} \leq \eta^{n-1}\left(\frac{t}{Kr}\|z_{\tau^{-n}\omega}\|_{s-1} + o(|t|)\right) \tag{5.3.8}$$

which can be rewritten, by taking the L^∞ norm

$$\|\boldsymbol{\pi}^n_u \boldsymbol{\phi} - \boldsymbol{\pi}^n_u(\boldsymbol{\phi} + t.\boldsymbol{z})\|_{L^\infty(\Omega,\mathcal{B}_{s-1})} \leq \eta^{n-1}\left(\frac{t}{Kr}\|\boldsymbol{z}\|_{L^\infty(\Omega,\mathcal{B}_{s-1})} + o(|t|)\right) \tag{5.3.9}$$

Dividing by t and taking the limit $t \to 0$, one obtains

$$\|\boldsymbol{Q}^{(0,1)}_{\boldsymbol{\pi}^n}(u, \boldsymbol{\phi})\|_{L^\infty(\Omega,\mathcal{B}_{s-1})} \leq \frac{\eta^{n-1}}{Kr} \tag{5.3.10}$$

This last estimate is valid whenever $\boldsymbol{\phi} \in L^\infty(\Omega, \mathcal{C}^*_{s-1})$ satisfies: there is $r > 0$, independent of $\omega \in \Omega$, such that $B_{s-1}(\phi_\omega, r) \subset \mathcal{C}^*_{s-1}$. In particular, it holds almost surely at $h_{\omega,u}$.

Introducing $\boldsymbol{Q}_u := \boldsymbol{Q}^{(0,1)}_{\boldsymbol{\pi}}(u, \boldsymbol{h}_u)$, one has by chain rule $\boldsymbol{Q}^n_u = \boldsymbol{Q}^{(0,1)}_{\boldsymbol{\pi}^n}(u, \boldsymbol{h}_u)$. Hence

$$\|\boldsymbol{Q}^n_u\|_{L^\infty(\Omega,\mathcal{B}_{s-1})} \leq \frac{\eta^{n-1}}{Kr} \tag{5.3.11}$$

and it follows that $\mathbb{1} - \boldsymbol{Q}_u$ is invertible on $L^\infty(\Omega, B_{s-1})$.

Note also that all the previous estimates are valid as soon as the cone contraction property of $\boldsymbol{\mathcal{L}}_u$ is satisfied, i.e $\mathbb{1} - \boldsymbol{Q}_u$ is bounded and invertible on $L^\infty(\Omega, B_t)$ if $\boldsymbol{\mathcal{L}}_u$ contracts a regular $\mathbb{R}$-cone in $\mathcal{B}_t$.

Therefore one can apply theorem 3.1 and obtain

$$\boldsymbol{h}_{u+v} - \boldsymbol{h}_u = (\mathbb{1} - \boldsymbol{Q}_u)^{-1}\boldsymbol{P}_u.v + o(v) \tag{5.3.12}$$

where

$$P_{\omega,u} = Q^{(1,0)}_{\pi_\omega}(u, h_\omega, u) = \frac{1}{p_{\omega,u}}\left[\partial_u \mathcal{L}_{\omega,u} h_{\omega,u} - \langle \ell, \partial_u \mathcal{L}_{\omega,u} h_{\omega,u}\rangle h_{\tau\omega,u}\right] \tag{5.3.13}$$

$$Q_{\omega,u} = Q^{(0,1)}_{\pi_\omega}(u, h_{\omega,u}) = \frac{1}{p_{\omega,u}}\left[\mathcal{L}_{\omega,u} - \langle \ell, \mathcal{L}_{\omega,u}\rangle h_{\tau\omega,u}\right] \tag{5.3.14}$$

Thus the differentiability of $u \in \mathcal{U} \mapsto \boldsymbol{h}_u \in L^\infty(\Omega, \mathcal{B}_{s-1})$ is proved.

For a fixed $u_0 \in \mathcal{U}$, $\mathcal{L}_{\omega,u_0} h_{\omega,u_0} \in \mathcal{C}^*_{s-1}$ almost surely, so that $\langle \ell_{s-1}, \mathcal{L}_{\omega,u} h_{\omega,u}\rangle > 0$ for u close enough of u_0.
Furthermore, one has

$$\begin{aligned}
\boldsymbol{p}_{u+v} &= \langle \ell_0, \boldsymbol{\mathcal{L}}_{u+v}\boldsymbol{h}_{u+v}\rangle && (5.3.15)\\
&= \boldsymbol{p}_u + \langle \ell_0, \boldsymbol{P}_0.v\rangle + \langle \ell_0, \boldsymbol{\mathcal{L}}_u[\boldsymbol{h}_{u+v} - \boldsymbol{h}_u]\rangle + (\|v\|_{\mathcal{B}} + \|\boldsymbol{h}_{u+v} - \boldsymbol{h}_u\|_{s-1})\,\epsilon(v, \|\boldsymbol{h}_{u+v} - \boldsymbol{h}_v\|_s) && (5.3.16)
\end{aligned}$$

By the results of the previous section, the error term $(\|v\|_{\mathcal{B}} + \|\boldsymbol{h}_{u+v} - \boldsymbol{h}_u\|_{s-1})\,\epsilon(v, \|\boldsymbol{h}_{u+v} - \boldsymbol{h}_v\|_s) = o(v)$ for $s < r$. Therefore, one can write

$$\frac{\boldsymbol{p}_{u+v}}{\boldsymbol{p}_u} = 1 + \frac{\langle \ell_0, \boldsymbol{P}_0.v\rangle}{\boldsymbol{p}_u} + \frac{\langle \ell_0, \boldsymbol{\mathcal{L}}_u[\boldsymbol{h}_{u+v} - \boldsymbol{h}_u]\rangle}{\boldsymbol{p}_u} + o(v) \tag{5.3.17}$$

$$\log\left(\frac{\boldsymbol{p}_{u+v}}{\boldsymbol{p}_u}\right) = \frac{\langle \ell_0, \boldsymbol{P}_0.v\rangle}{\boldsymbol{p}_u} + \frac{\langle \ell_0, \boldsymbol{\mathcal{L}}_u[\boldsymbol{h}_{u+v} - \boldsymbol{h}_u]\rangle}{\boldsymbol{p}_u} + o(v) \tag{5.3.18}$$

From this last estimate and (5.3.12), one obtains the following "Taylor expansion" at order one for the characteristic exponent map $\in \mathcal{U} \mapsto \chi_u$

$$\chi_{u+v} - \chi_u = \int_\Omega \frac{\langle \ell_0, \boldsymbol{P}_0.v + \boldsymbol{\mathcal{L}}_u[\mathbb{1} - \boldsymbol{Q}_u]^{-1}\boldsymbol{P}_u.v\rangle}{\boldsymbol{p}_u} d\mathbb{P} + o(v) \tag{5.3.19}$$

Twice differentiability As noticed previously, the invertibility of $\mathbb{1} - \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,1)}(u, \boldsymbol{h}_u)$ on $\mathcal{B}_t$ remains as soon as the cone $\mathcal{C}_t$ is contracted by $\mathcal{L}_{\omega,u}$. For $t = s-2$, we write the Taylor expansion at $u \in \mathcal{U}$:

$$\begin{aligned}\boldsymbol{h}_{s-2,u+v} - \boldsymbol{h}_{s-2,u} &= \boldsymbol{\pi}_{s-2,u+v}\boldsymbol{h}_{s-2,u+v} - \boldsymbol{\pi}_{s-2,u}\boldsymbol{h}_{s-2,u} \\ &= \boldsymbol{Q}_{\boldsymbol{\pi}}^{(1,0)}(u, \boldsymbol{h}_{s-1,u}).v + \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,1)}(u, \boldsymbol{h}_{s-1,u})[h_{s-2,u+v} - \boldsymbol{h}_{s-2,u}] \\ &+ \boldsymbol{Q}_{\boldsymbol{\pi}}^{(2,0)}(u, \boldsymbol{h}_{s,u}).[v] + \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,2)}(u, \boldsymbol{h}_{s,u}).[\boldsymbol{h}_{s-1,u+v} - \boldsymbol{h}_{s-1,u}] \\ &+ \boldsymbol{Q}_{\boldsymbol{\pi}}^{(1,1)}(u, \boldsymbol{h}_{s,u}).[v, \boldsymbol{h}_{s-1,u+v} - \boldsymbol{h}_{s-1,u}] + \boldsymbol{R}_2(v, \boldsymbol{h}_{s,u+v} - \boldsymbol{h}_{s,u})\end{aligned} \tag{5.3.20}$$

Using differentiability of $u \in \mathcal{U} \mapsto \boldsymbol{h}_{s-1,u}$, and taking only terms of order at most $\|v\|_{\mathcal{B}}^2$, we can write

$$\begin{aligned}&\left[\mathbb{1} - \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,1)}(u, \boldsymbol{h}_{s-1,u})\right](\boldsymbol{h}_{s-2,u+v} - \boldsymbol{h}_{s-2,u}) \\ &= \boldsymbol{Q}_{\boldsymbol{\pi}}^{(1,0)}(u, \boldsymbol{h}_{s-1,u}).v + \boldsymbol{Q}_{\boldsymbol{\pi}}^{(2,0)}(u, \boldsymbol{h}_{s,u}).[v] + \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,2)}(u, \boldsymbol{h}_{s,u})[D_u\boldsymbol{h}_{s-1,u}.v] \\ &+ \boldsymbol{Q}_{\boldsymbol{\pi}}^{(1,1)}(u, \boldsymbol{h}_{s,u}).[v, [D_u\boldsymbol{h}_{s-1,u}.v] + o(\|v\|_{\mathcal{B}}^2)\end{aligned} \tag{5.3.21}$$

the $o(\|v\|^2)$ term coming from bi-linearity of $v \in \mathcal{B} \mapsto \boldsymbol{Q}^{i,j}(.,.)[v]$ and the assumption on the error term $\boldsymbol{\mathcal{R}}(v,.)$
With the invertibility of $\mathbb{1} - \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,1)}(u, \boldsymbol{h}_u)$ on $\mathcal{B}_{s-2}$, this yields

$$\begin{aligned}\boldsymbol{h}_{s-2,u+v} - \boldsymbol{h}_{s-2,u} &= \left[\mathbb{1} - \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,1)}(u, \boldsymbol{h}_{s-1,u})\right]^{-1}\boldsymbol{Q}_{\boldsymbol{\pi}}^{(1,0)}(u, \boldsymbol{h}_{s-1,u}).v \\ &+ \left[\mathbb{1} - \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,1)}(u, \boldsymbol{h}_{s-1,u})\right]^{-1}\left(\boldsymbol{Q}_{\boldsymbol{\pi}}^{(2,0)}(u, \boldsymbol{h}_{s,u}).[v] + \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,2)}(u, \boldsymbol{h}_{s,u})[D_u\boldsymbol{h}_{s-1,u}.v]\right) \\ &+ \left[\mathbb{1} - \boldsymbol{Q}_{\boldsymbol{\pi}}^{(0,1)}(u, \boldsymbol{h}_{s-1,u})\right]^{-1}\left(\boldsymbol{Q}_{\boldsymbol{\pi}}^{(1,1)}(u, \boldsymbol{h}_{s,u}).[v, D_u\boldsymbol{h}_{s-1,u}.v]\right) + o(\|v\|_{\mathcal{B}}^2)\end{aligned} \tag{5.3.22}$$

which is the announced Taylor expansion of order 2.

We now turn to the similar expansion for $u \in \mathcal{U} \mapsto \chi_u$. We write

$$\begin{aligned}&\boldsymbol{\mathcal{L}}_{u+v}\boldsymbol{h}_{s-2,u+v} - \boldsymbol{\mathcal{L}}_u\boldsymbol{h}_{s-2,u} = \\ &\boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(1,0)}(u,\boldsymbol{h}_{s-1,u}).v + \boldsymbol{\mathcal{L}}_u.[\boldsymbol{h}_{s-2,u+v} - \boldsymbol{h}_{s-2,u}] \\ &+ \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(2,0)}(u,\boldsymbol{h}_{s,u}).v + \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(0,2)}(u,\boldsymbol{h}_{s,u}).[\boldsymbol{h}_{s-1,u+v} - \boldsymbol{h}_{s-1,u}] + \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(1,1)}(u,\boldsymbol{h}_{s,u})[v,\boldsymbol{h}_{s-1,u+v} - \boldsymbol{h}_{s-1,u}] \\ &+ \boldsymbol{R}(v,\boldsymbol{h}_{s,u+v} - \boldsymbol{h}_{s,u})\end{aligned} \tag{5.3.23}$$

Using the previously established regularity results on $u \in \mathcal{U} \mapsto \boldsymbol{h}_{s-1,u}$, and taking the product against ℓ, we get

$$\begin{aligned}&\boldsymbol{p}_{u+v} - \boldsymbol{p}_u = \langle \ell, \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(1,0)}(u,\boldsymbol{h}_{s-1,u}).v + \boldsymbol{\mathcal{L}}_u[D_u\boldsymbol{h}_{s-1,u}.v]\rangle \\ &\quad + \langle \ell, \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(2,0)}(u,\boldsymbol{h}_{s,u}).v\rangle + \langle \ell, \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(0,2)}(u,\boldsymbol{h}_{s,u}).[D_u\boldsymbol{h}_{s-1,u}.v]\rangle \\ &\quad + \langle \ell, \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(1,1)}(u,\boldsymbol{h}_{s,u})[v, D_u\boldsymbol{h}_{s-1,u}.v]\rangle + \langle \ell, \boldsymbol{\mathcal{L}}_u.D_u^2\boldsymbol{h}_{s-2,u}.v\rangle + o(\|v\|_{\mathcal{B}}^2)\end{aligned} \tag{5.3.24}$$

Dividing by $\boldsymbol{p}_u > 0$, and taking the integral over $(\Omega, \mathbb{P})$ one has

$$\begin{aligned}&\chi_{u+v} - \chi_u = \int_\Omega \log\left(\frac{\boldsymbol{p}_{u+v}}{\boldsymbol{p}_u}\right) d\mathbb{P} \\ &= \int_\Omega \frac{1}{\boldsymbol{p}_u}\langle \ell, \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(1,0)}(u,\boldsymbol{h}_{s-1,u}).v + \boldsymbol{\mathcal{L}}_u[D_u\boldsymbol{h}_{s-1,u}.v]\rangle d\mathbb{P} \\ &+ \int_\Omega \frac{1}{\boldsymbol{p}_u}\langle \ell, \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(2,0)}(u,\boldsymbol{h}_{s,u}).v + \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(0,2)}(u,\boldsymbol{h}_{s,u}).[D_u\boldsymbol{h}_{s-1,u}.v]\rangle d\mathbb{P} \\ &+ \int_\Omega \frac{1}{\boldsymbol{p}_u}\langle \ell, \boldsymbol{Q}_{\boldsymbol{\mathcal{L}}}^{(1,1)}(u,\boldsymbol{h}_{s,u})[v, D_u\boldsymbol{h}_{s-1,u}.v] + \boldsymbol{\mathcal{L}}_u D_u^2\boldsymbol{h}_{s-2,u}.v\rangle d\mathbb{P} + o(\|v\|_{\mathcal{B}}^2)\end{aligned} \tag{5.3.25}$$

which is the announced Taylor expansion at order 2. □

Chapter 6

Conclusion and further research

6.1 Linear response for Anosov systems

In the latter parts, we have shown a variety of regularity (w.r.t parameters) results for *quantities of interest* in the thermodynamic formalism of expanding maps: topological pressure and entropy, Gibbs measures, variance in the central limit theorem, rate of mixing, rate of large deviations.

However, we have to acknowledge that the scope of those results remains limited, as it is well known that expanding systems are but a toy model to the true mathematical models of chaotic dynamics. In that spirit, it would be interesting to extend our results Anosov or Axiom A diffeomorphisms. In what follows, we would like to discuss the possible generalizations of our results to the hyperbolic case, and even more generalizations based on the same fixed-point based method we developed.

Let's briefly recall the definition of an Anosov system : Let $T : M \to M$ be a C^r, $r \geq 1$ map of a Riemann manifold such that

- There exists $\lambda > 1$, called the hyperbolicity factor of T, such that at every point $x \in M$ there is a decomposition of the tangent space in invariant sub-bundles $T_xM = E_s(x) \oplus E_u(x)$, where

$$\forall\ v \in E_s(x), \|DT^n(x).v\| \leq C\lambda^n\|v\| \tag{6.1.1}$$

$$\forall\ v \in E_u(x), \|DT^{-n}(x).v\| \leq C\lambda^{-n}\|v\| \tag{6.1.2}$$

- T is topologically transitive, i.e there is a dense orbit.

Among those systems, one can give the example of the endomorphism induced on $\mathbb{T}^2$ by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, the famous *Arnold cat map.*

Very broadly speaking, what one should remember from the last part is that, in order to prove regularity with respect to the parameters of *quantities of interest*, one needs two majors ingredients:

1. A *spectral gap* for the transfer operator, on a suitable scale of Banach spaces. This gives us the existence of the fixed point (assumption (i) in theorem 3.1), and furthermore plays a

key role in obtaining invertibility of the partial differential (assumption (iii)). In the setting of expanding maps, Ruelle theorem tells us that such a gap exists on the scale $(C^r)_{r>0}$.

2. Ad hoc estimates on the composition operator, acting on the space where the spectral gap was previously obtained. This is the key point to obtain Hölder and Lipschitz continuity of the transfer operator with respect to the parameters (once seen as acting on suitable spaces), as well as our differentiability condition. In the context of expanding maps, this derives from our parameter-wise adaptation of de la Llave and Obaya regularity estimates on the composition operator acting on Hölder spaces.

With those two ingredients in mind, the question of whether one can generalize our approach to hyperbolic systems boils down to the two following questions:

1. Can one construct a Banach space on which the (weighted) transfer operator associated to an Anosov/Axiom A system admits a spectral gap ?

2. On the aforementioned spaces, can one establish regularity estimates for the composition operator ?

Fortunately, the answer to both those questions is yes ! However, one should not be too naive: the following result show that such a space cannot be a usual space of functions:

Proposition 6.1
Let T be a C^∞, volume preserving Anosov diffeomorphism with hyperbolicity factor $\lambda > 1$. Let $\mathcal{L}_T : f \mapsto f \circ T^{-1}$ be its transfer operator. Then its essential spectral radius on C^k is $\geq \lambda^k$.

A great amount of work was devoted to construct so-called *anisotropic Banach spaces*, on which the transfer operator $\mathcal{L}$ associated to an hyperbolic map T admits a spectral gap, starting with the seminal papers by Gouëzel-Liverani [39, 40] (that we already mentioned, due to its development of weak spectral perturbation theory. It is indeed a seminal paper...) and Baladi-Tsujii [8].
Since then, a variety of such spaces have been constructed, in semi-classical analysis (see, e.g [24]), in the study of smooth Anosov flows ([11, 34]), or other hyperbolic flows (e.g, Morse-Smale flows in [Dang-Rivière]).

We will now lay a sort of blueprint for constructing those spaces, by explaining the properties we expect them to have. Let $\mathcal{B}$ be an "anisotropic Banach space".

1. The transfer operator $\mathcal{L}_T$ of an hyperbolic map should have a spectral gap when acting on $\mathcal{B}$: it is after all, the main motivation !

2. It follows from Gouëzel result (see prop 6.1) that $\mathcal{B}$ cannot be a classical function space. In fact, it was already acknowledged by Rugh in the mid 90's [Rugh94] that one should look for *distribution spaces* instead. It should however contain smooth observables. Therefore, we want that
$$C^r \subset \mathcal{B} \subset (C^r)^*$$

3. It should behave as a function space in the unstable direction, and as a distribution space in the stable direction.
Indeed, we at least want $\mathcal{L}_T$ to mimic the spectral behavior of a transfer operator associated to an expanding map when the observable is only defined on the unstable manifold.

This informal discussion is helpful for visualizing what one expects from such spaces, but it does not tell us how to effectively construct one. Such a construction is too long for us to expose here, and instead we refer to the seminal papers [39, 8] for the details.

6.2 Limit theorem for random product of expanding maps

In [18], the authors introduce a generalization to random dynamical systems of the well known *Nagaev-Guivarc'h* spectral method, a powerful approach to establish limit theorems for (autonomous) dynamical systems and Markov chains. Let us briefly recall how it operates, following Gouëzel review [38].

Very broadly speaking, the idea is to follow the elementary proof of the central limit theorem for L^2, i.i.d random variables.
Given a Lipschitz continuous dynamical system T acting on some metric space (M, d), we let $f \in C^\alpha(M)$ ($\alpha \in [0,1]$) be an observable, and consider the random process $(f \circ T^n)_n$. We would like to study the (probabilistic) asymptotic behavior of the Birkhoff sums

$$S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$$

In that endeavor, we introduce the *twisted transfer operator*, acting on $C^\alpha(M)$ by

$$\mathcal{L}_t\phi := \mathcal{L}_T(e^{itf}\phi) \tag{6.2.1}$$

where $\mathcal{L}_T$ is the Ruelle transfer operator (2.2.7). Now, the proof relies on the following steps:

1. Represent $\mathbb{E}\left(e^{iS_n(f)}\right)$ as $\int_M \mathcal{L}_t^n \mathbf{1} dm$
2. Establishing that $\mathcal{L}_t$ admits a spectral gap on $C^\alpha(M)$, with maximal eigenvalue $\lambda(t)$.
3. The map $t \mapsto \lambda(t)$ is C^2

In our setting, the first point is a straightforward consequence of the duality property (2.2.9), the second and third follows from the spectral gap of $\mathcal{L}_T$ on C^α and standard perturbation theory (note that $\mathcal{L}_t$ is an analytic perturbation of $\mathcal{L}_T$!). Here no loss of regularity happens, as one perturbs only the weight and not the dynamic itself.

The steps explained here can be used to show a wide variety of limit theorem for dynamical systems: we refer to [38] for more details on the subject.

The generalization of [18] goes along the same lines, in a random context: given $(\Omega, \mathcal{F}, \mathbb{P})$ a probabilistic space, $\tau : \Omega \to \Omega$ invertible and ergodic, we let $T \in \mathcal{M}(\Omega, C^\alpha([0,1]))$ be a random dynamical system over (Ω, τ). We want to establish (quenched) limit theorem for processes of the type $(f_\omega \circ T_\omega^{(n)})_{n\geq 0}$ where $f : \Omega \to BV([0,1])$ is measurable.

For that purpose, the paper [18] introduce a *twisted transfer operator* $\mathcal{L}_{\omega,t}$ defined by:

$$\mathcal{L}_{\omega,t}\phi(x) := \mathcal{L}_\omega\left(e^{tf_\omega}\phi\right) \tag{6.2.2}$$

and suggests the following random analogue to the Nagaev-Guivarc'h method:

1. Represent the quenched Birkhoff sums $S_n(f_\omega) := \sum_{k=0}^{n-1} f_\omega \circ T_\omega^{(k)}$ as an integral involving the n-th random product of twisted transfer operator.

2. Establish *quasi-compactness* of the transfer operator cocycle $\mathcal{L}_{\omega,t}$ for t small enough.

3. Show regularity w.r.t the parameter of the top characteristic exponent and associated Oseledets space of the transfer operator cocycle.

The first item is usually straightforward, and we refer to the paper [18] for adequate explanations and proofs.

Where I believe that it is possible to both extend and simplify the original approach concerns the last two steps. This relies mostly on the use of the theory of complex cone contraction: indeed, the twisted transfer operator associated e.g to some uniformly expanding map or some tent maps is a strict and uniform contraction of a regular complex cone, which is the canonical complexification of (5.2.1) in the first case, and the canonical complexification of the cone used in [56]:

$$\mathcal{C}_a := \{\phi \in BV([0,1]), \phi \geq 0, Var_{[0,1]}(\phi) \leq a \int_0^1 \phi dm\}$$

Building on works of Dubois and Rugh[71, 21, 22] one may establish a complex analog to [20, Theorem 3.1], thus establishing a gap in the Oseledets-Lyapunov spectrum of the twisted transfer operator cocycle.

As for the third step, one may directly use theorem 5.3 and the bound (5.2.41), to establish [71, Theorem 10.2] on the analyticity of the top characteristic exponent in this context.

Now this contrast greatly with the the techniques used in [18], where a great deal of effort goes into showing twice differentiability of the top characteristic exponent w.r.t the parameter. Here not only does this regularity comes in a simpler way, but it is a much stronger regularity: analyticity vs. twice differentiability. Therefore not only can we recover all the limit theorem of [18], but it might (this is at the moment just a conjecture) also be possible to establish finer results, such as Berry-Essen estimates, in the spirit of [22].

6.3 Response for systems with additive noise

This project is an ongoing collaboration with Stefano Galatolo. It is similarly concerned with a problem of response, linear or higher-order, but in a different context: instead of looking at maps in the expanding class, or even with the expansiveness property (1.0.1), we will look at deterministic systems perturbed by additive noise. In this context, we make **no hyperbolicity assumption** on the deterministic part of the dynamic, and instead exploit the effects of additive noise on the resulting dynamic: mixing and regularization.

More precisely, we are interested in systems of the following form

$$T_{\xi,\delta}(x) = T_\delta(x) + X_\xi \tag{6.3.1}$$

where $T_\delta : [0,1] \to [0,1]$ are maps of the interval, or of the circle $\mathbb{S}^1$ and X_ξ is a random perturbation, given by a density kernel ρ_ξ.

We study two major cases:

1. The case of a uniformly distributed noise: $\rho_\xi = \frac{1}{\xi}\mathbb{1}_{[-\xi/2,\xi/2]} \in BV([0,1])$.

2. The case of a Gaussian noise: $\rho_\xi = e^{-x^2/2\xi^2} \in C_0^\infty(\mathbb{R})$.

At the level of transfer operator $\mathcal{L}_{\delta,\xi}$, acting on (say) L^1 functions of the interval, this translates to

$$\mathcal{L}_{\delta,\xi}\phi := \rho_\xi * \mathcal{L}_{T_\delta}\phi \tag{6.3.2}$$

where $\mathcal{L}_{T_\delta}$ is the transfer operator (2.2.7) of the map T_δ, and $*$ is the convolution product.

Using the Krylov-Bogolubov procedure, one can construct an invariant measure $\mu_{\delta,\xi} \in L^1([0,1])$ for the system, which is a fixed-point for the transfer operator. We are interested in the question of the regularity of the map

$$(\delta,\xi) \mapsto \mu_{\delta,\xi} \in L^1([0,1]) \tag{6.3.3}$$

Those type of question has already been partially studied in a recent paper by Giulietti and Galatolo [30], where they studied the question of C^1 regularity of the aforementioned map (6.3.3), and obtained a linear response formula, both for changes in the deterministic parameter δ and variations in the size of the random perturbation ξ.

More precisely, they based their approach on the following abstract linear response statement in the presence of mixing and regularization, in the spaces $BV([0,1])$, $L^1([0,1])$ and $BS([0,1])$ the space of signed Borel measures endowed with the norm

$$\|\mu\|_W := \sup_{\|g\|_{Lip}\leq 1} \left| \int_0^1 g d\mu \right|$$

Theorem 6.1 ([30], Thm 3)
Let $(\mathcal{L}_\delta)_{\delta\in[0,\epsilon]}$ be a family of bounded Markov operators acting on $BV([0,1])$.
Assume furthermore that

1. *For every $\delta \in [0,\epsilon]$, there is a $h_\delta \in BV$, such that $\mathcal{L}_\delta h_\delta = h_\delta$ and $\|h_\delta\|_{BV} \leq C$ for some $C > 0$ independent of δ.*

2. *For any $\phi \in V_1 := \{\phi \in L^1,\ \int_0^1 \phi(x)dx = 0\}$, $\|\mathcal{L}_0^n\phi\|_{L^1} \underset{n\to+\infty}{\longrightarrow} 0$*

3. *The unperturbed operator is regularizing from W to L^1 and from L^1 to BV, i.e $\mathcal{L}_0 : W \to L^1$ and $\mathcal{L}_0 : L^1 \to BV$ is bounded.*

4. *$\mathcal{L}_\delta$ is a relatively continuous perturbation of $\mathcal{L}_0$, i.e $\|\mathcal{L}_0 - \mathcal{L}_\delta\|_{BV\to L^1} \leq C\delta$ and similarly $\|\mathcal{L}_0 - \mathcal{L}_\delta\|_{L^1\to W} \leq C\delta$, and there exists a* derivative operator *$\dot{L}h_0 \in V_0$, where $V_0 := \overline{V_1}^W$ such that*

$$\left\| \frac{1}{\delta}(\mathcal{L}_0 - \mathcal{L}_\delta)h_0 - \dot{L}h_0 \right\|_W \underset{\delta\to 0}{\longrightarrow} 0 \tag{6.3.4}$$

Then one has the following: $\mathcal{R}(z,\mathcal{L}_0): V_0 \to W$ *is a bounded operator, and*

$$\left\| \frac{h_\delta - h_0}{\delta} - \mathcal{R}(1,\mathcal{L}_0)\dot{L}h_0 \right\|_W \underset{\delta\to 0}{\longrightarrow} 0 \tag{6.3.5}$$

i.e $\delta \in [0,\epsilon] \mapsto h_\delta \in W$ *is* C^1 *at* $\delta = 0$ *and* $\mathcal{R}(1,\mathcal{L}_0)\dot{L}h_0$ *is the first-order term in the variation of equilibrium measure in the family* $\mathcal{L}_\delta$.

Our approach to study the mixing properties of such systems relies on *computer-assisted proofs*, i.e the use of a computer certified estimate to establish decay of the L^1 norm of the unperturbed system $T_{0,\xi} = T_0 + X_\xi$.

Using an *Ulam method*, one can construct a finite dimensional matrix approximating the transfer operator in the L^1 norm. Calling $\mathcal{L}_{\epsilon,\xi}$ this finite dimensional (of order roughly $\frac{1}{\epsilon}$) approximation, we remark that it is also a weak contraction in the L^1 norm: the computer will thus be able to find a couple (N,α) such that $\|\mathcal{L}_{\epsilon,\xi}^N\phi\|_{L^1} \leq \alpha < 1$ for every $\phi \in L^1$, $\int_0^1 \phi dm = 0$. For more details on the inner workings of the algorithm, the precise estimates and motivations behind it, we refer to [31, §3-4].

The fact that the additive noise produces mixing tells us that it will be an effective replacement to hyperbolicity assumptions, allowing us to use the functional method based on the transfer operator (2.2.7) spectral properties.

More particular to this situation is the effect of regularization, which will counter the loss of regularity effect seen at play in deterministic systems or for general random products. Heuristically, it can be understood as an averaging effect of randomness. Analytically, we already see where the regularizing effect on (6.3.2), the regularity of a convolution product being the regularity of the smoother term.

In particular, this yields that $\mathcal{L}_{\delta,\xi} : L^1([0,1]) \to BV([0,1])$ (resp $\mathcal{L}_{\delta,\xi} : L^1([0,1]) \to C_0^\infty(\mathbb{R})$)is a bounded operator.

We plan to generalize this statement to the case of several derivatives and higher-order response. This imply to find a proper functional framework to formulate this regularity: the scale of Banach spaces $W^{k,1} \hookrightarrow L^1 \hookrightarrow W^{-k,1}$ with $W^{k,1}$ being the classical Sobolev space on $[0,1]$ and $W^{-k,1}$ the space of k-distributional derivatives of L^1 functions appears to be a natural candidate in this endeavor.

We also need to find the proper generalization for the definition of the "derivatives operators" and the kind of convergence towards it: it would seems that one needs some type of uniform convergence for the smaller derivatives, when a point-wise convergence works for the higher-order derivative.

The applications we have in mind concerns several models of physical and mathematical interest: regularity for the Lyapunov exponent of a model of the Belusov Zhabotinski reaction, response for tent maps in presence of noise, and particularly giving a rigorous proof of a phenomena numerically observed by physicists [13] [DiGarbo, Private communication]: regularization of the rotation number of Arnold standard circle map in the presence of noise. Indeed, it is well known that the map "rotation number vs driving frequency" is a natural example of devil's staircase, i.e of a map that is both non constant and with derivative zero on a dense set. It was observed in numerical experiments that when noise is added (Gaussian or uniformly distributed), this map becomes a smooth one.

Other questions of interest to us include:

- The control problem for the invariant measure, as was sketched in [32].
- The case of systems with noise depending on the point, as it is very natural in many applications.
- The "zero-noise" limit, where the regularizing effect is lost.

Appendix A

Cone contraction theory: Birkhoff cones and $\mathbb{C}$-cones

In this section, E will denote a real Banach space.

A.1 Properties of Birkhoff cones

Definition A.1
Let $\mathcal{C} \subset E$. We say that $\mathcal{C}$ is a closed convex cone if

- $\mathbb{R}_+\mathcal{C} = \mathcal{C}$, *i.e $\mathcal{C}$ is stable by multiplication with a positive scalar.*
- $\mathcal{C}$ *is a closed and convex subset of E.*

We will say that the cone is ***proper*** *if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$. We also define the* dual *of the real closed convex cone $\mathcal{C}$ to be the set of non-zero functionals on $\mathcal{C}$, i.e*

$$\mathcal{C}' := \{m \in E', \langle m, x\rangle \neq 0 \ \forall x \in \mathcal{C}^*\} \tag{A.1.1}$$

where $\mathcal{C}^* = \mathcal{C}\backslash\{0\}$.

Definition A.2
Let $\mathcal{C} \subset E$ be a closed, convex cone. We say that $\mathcal{C}$ is

1. ***inner regular*** *if there exists $x \in \mathcal{C}^*$, $\rho > 0$ such that $B_E(x, \rho) \subset \mathcal{C}$, i.e $\mathcal{C}$ has non-empty interior in E.*
2. *Let $m \in E'$ be non zero, we define the* ***aperture of $\mathcal{C}$ relative to m*** *by*

$$K(\mathcal{C}, m) = \sup_{u \in \mathcal{C}^*} \frac{\|m\|.\|u\|}{\langle m, u\rangle} \in [1, +\infty] \tag{A.1.2}$$

   We also define the aperture of $\mathcal{C}$ to be $K(\mathcal{C}) = \inf_{\substack{m \in E' \\ m \neq 0}} K(\mathcal{C}, m)$.
   We say that $\mathcal{C}$ is ***outer regular*** *(or of K-bounded aperture) if $K(\mathcal{C}) < +\infty$.*

3. *We say that $\mathcal{C}$ is of* ***K-bounded sectional aperture*** *if for every $x, y \in \mathcal{C}$, the sub-cone $Span(x, y) \cap \mathcal{C}$ is of K-bounded aperture.*

4. *A cone is said to be* ***reproducing*** *if there exists $g < +\infty$ such that every $x \in E$ decomposes into $x = x_1 - x_2$, with $x_1, x_2 \in \mathcal{C}$ and $\|x_1\| + \|x_2\| \leq g\|x\|$.*

We will call a closed, convex proper cone a $\mathbb{R}$-cone (or a Birkhoff cone). We will say that $\mathcal{C}$ is ***regular*** *if it is both inner and outer regular.*

We also recall the definition of the **Hilbert metric**:

Definition A.3
Let $\mathcal{C} \subset E$ be a closed, convex, proper cone, and let $x, y \in \mathcal{C}^*$. We define

$$\delta(x,y) = \inf\{t > 0,\ tx - y \in \mathcal{C}\} \tag{A.1.3}$$

$$d_{\mathcal{C}}(x,y) = \log(\delta(x,y)\delta(y,x)) \tag{A.1.4}$$

Then $d_{\mathcal{C}}$ is a projective distance, called the ***Hilbert metric****.*

Lemma A.1
Let $\mathcal{C} \subset E$ be a regular $\mathbb{R}$-cone, and let $d_{\mathcal{C}}$ be the associated Hilbert metric. If there exists $r > 0$ such that $B(x,r) \subset \mathcal{C}^*$, then for every $y \in B(x,r)$

$$d_{\mathcal{C}}(x,y) \leq \log\left(\frac{r + \|y-x\|}{r - \|y-x\|}\right) \tag{A.1.5}$$

In the applications, it is of foremost importance to be able to compare the Hilbert metric with the distance given by the norm of the ambient Banach space. It is the object of this next lemma:

Lemma A.2
Let $\mathcal{C} \subset E$ be a real, proper convex closed cone. Assume furthermore that $\mathcal{C}$ is of K bounded sectional aperture. Then for all $x, y \in \mathcal{C}^*$, letting $m \in E'$ be the non zero linear form given by the bounded sectional aperture property, one has

$$\left\| \frac{x}{\langle m, x\rangle} - \frac{y}{\langle m, y\rangle} \right\|_E \leq \frac{1}{2K} d_{\mathcal{C}}(x,y) \tag{A.1.6}$$

We can now state the main result of this section: a linear operator sending a cone inside another shrinks the Hilbert metric.

Theorem A.1 (Birkhoff theorem)
Let E_1, E_2 be Banach spaces, and $\mathcal{C}_1 \subset E_1$, $\mathcal{C}_2 \subset E_2$ be proper, closed convex cones. Let $\mathcal{L} : E_1 \to E_2$ be a linear map, such that $\mathcal{L}(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$. Let $\Delta = diam_{\mathcal{C}_2}(\mathcal{L}(\mathcal{C}_1^*)) \in [0, +\infty]$. Then one has

$$d_{\mathcal{C}_2}(\mathcal{L}x, \mathcal{L}y) \leq \tanh\left(\frac{\Delta}{4}\right) d_{\mathcal{C}_1}(x,y) \tag{A.1.7}$$

This result has important consequences for the study of spectral properties of cone-preserving operators: we will state two of them.

First, a quasi-compact operator (in the sense of §2.4) preserving a Birkhoff cone has a spectral gap: this is the content of the *Krein-Rutman theorem*:

Theorem A.2 ([51], Theorem 6.3)
Let $\mathcal{C}$ be an inner regular Birkhoff cone of a real Banach space E. Let $\mathcal{L} : E \to E$ be a quasi-compact operator, such that $\mathcal{L}(\mathcal{C}^*) \subset Int(\mathcal{C})$. Then $\mathcal{L}$ admits a spectral gap.

Second, it is noteworthy that the conditions on the spectrum of the cone-contracting operator can be replaced by geometrical assumptions on the cone: a linear operator preserving a reproducing Birkhoff cone with bounded sectional aperture (e.g, if the cone is regular) has a *spectral gap*, without condition on its spectrum: this a theorem of Birkhoff [9].

Theorem A.3
Let $\mathcal{C} \subset E$ be a reproducing Birkhoff cone with bounded sectional aperture. We assume that $\mathcal{L} : E \to E$ is a strict cone contraction, i.e $\mathcal{L}(\mathcal{C}^*) \subset \mathcal{C}^*$ and $diam_{\mathcal{C}}(\mathcal{L}(\mathcal{C}^*)) < +\infty$. Then $\mathcal{L}$ has a spectral gap.

We end this section with an intuitive and useful lemma; as it is not clearly stated in any form to the best of our knowledge, we provide a proof here.

Lemma A.3
Let $\mathcal{C}_1 \subset \mathcal{C}_2$ be two regular Birkhoff cones, such that $\Delta := diam_{\mathcal{C}_2}\mathcal{C}_1^* < +\infty$. Then there exists a $\rho > 0$, such that for every $v \in \mathcal{C}_1$, $B_E(v, \rho\|v\|_E) \subset \mathcal{C}_2$.

Proof. As $\mathcal{C}_1$ is a regular Birkhoff cone, in particular it has non-empty interior in E, i.e there exists $r > 0$, and $a \in \mathcal{C}_1$, with norm 1 such that $B_E(a, r) \subset \mathcal{C}_1$. Let $v \in \mathcal{C}_1$. As $\mathcal{C}_1$ has finite diameter for the Hilbert metric $d_{\mathcal{C}_2}$, it follows that the projective distance between v and a is finite: $d_{\mathcal{C}_2}(v, a) \leq \Delta < +\infty$. We claim that this implies $B(v, \frac{re^{-\Delta}}{K}\|v\|_E) \subset \mathcal{C}_2$, where K is the sectional aperture of $\mathcal{C}_2$.

Indeed, start by remarking that if $B(a, r) \subset \mathcal{C}_1$, then for any $v \in \mathcal{C}_1^*$, $B(v, r\delta_2(v, a)^{-1}) \subset \mathcal{C}_2$. This is obvious when $\delta(v, a) = +\infty$, so that we can assume $\delta(v, a)$ to be finite. Note that one can replace a and r by λa and λr, and by definition, $\delta(v, \lambda a) = \lambda\delta(v, a)$ for all $\lambda > 0$, so that one can assume that $\delta(v, a) > 1$.
By closure of $\mathcal{C}_2$, $t = \delta_2(v, a)$ satisfies $tv - a \in \mathcal{C}_2$, and stability by multiplication with a positive scalar implies that $b = \frac{1}{t-1}(tv - a) \in \mathcal{C}_2$. Now let $u \in E$, $\|u\|_E < r$, then $a + u \in B(a, r) \subset \mathcal{C}_1$. In particular, one has

$$v + \frac{1}{t}u = \frac{t-1}{t}b + \frac{1}{t}(a + u) \in \mathcal{C}_2$$

for every $u \in B_E(0, r)$. It follows that $B_E(v, r\delta(v, a)^{-1}) := \{v + \frac{1}{t}u,\ \|u\|_E < r\} \subset \mathcal{C}_2$.

It is clear from the definition that $d_{\mathcal{C}_2}(v, a) \leq \Delta$ implies that $\delta_2(v, a)^{-1} \geq e^{-\Delta}\delta_2(a, v)$. Now, we show that if $\mathcal{C}_2$ has K-bounded sectional aperture, $K \geq 1$, then $\delta_2(u, v) \geq \frac{\|v\|_E}{K\|u\|_E}$.
It is obvious if $\delta_2(u, v) = \infty$, so we can assume $t = \delta_2(u, v)$ is finite. Let $s := \inf\{\lambda \in \mathbb{R},\ \lambda u + v \in \mathcal{C}_2\}$. Note that $s \neq -\infty$, because if it were, then one would have $-u + v/n \in \mathcal{C}_2$ for every $n \geq 1$, which would imply that $-u \in \mathcal{C}_2$: as $\mathcal{C}_2$ is proper, this is impossible. Thus, $a = su + v \in \mathcal{C}_2$, and $b = tu - v \in \mathcal{C}_2$. $tu + v = b + 2v \in \mathcal{C}_2$, and thus $t \geq s$. Furthermore, $(t + s)u = a + b \in \mathcal{C}_2$, so that $t + s \geq 0$ by properness of $\mathcal{C}_2$. Finally,

$$\|2v - (t - s)u\| = \|a - b\| \leq \|a\| + \|b\| \leq \langle \ell, a\rangle + \langle \ell, b\rangle \leq K\|a + b\| = K(t + s)\|u\|$$

by taking an outer regularity form ℓ of norm 1 for $\mathcal{C}_2$. This yields

$$\|v\| \leq K\|u\|\frac{t+s}{2} + \|u\|\frac{t-s}{2} \leq Kt\|u\|$$

which establishes the claim.
This proves the lemma, with $\rho = K^{-1}re^{-\Delta}$. □

In this section, we are interested in generalizing the previous results to the setting of complex valued operators, acting on a complex Banach spaces E. The interest of such a generalization comes from certain class of perturbative problems, where one naturally consider the dependency on (small) complex parameters (e.g, to recover the Gibbs measure, or the variance in the central limit theorem, from the topological pressure, see § 4.4 and theorem 4.5.8)

Unfortunately, the notion of positivity, or more generally the partial ordering induced by a proper Birkhoff cone, which is crucial to define both the cones themselves and the Hilbert metric, is now gone. It was Rugh's idea [71] (also followed by Dubois) to replace the Hilbert metric with a *complex gauge*. What follows is taken from [71].

Definition A.4
Let E be a complex Banach space.

- *We say that $\mathcal{C} \subset E$ is a* ***closed complex cone*** *if it is a closed subset of E, $\mathbb{C}$-invariant and $\mathcal{C} \neq \{0\}$.*
- *We say that the closed complex cone $\mathcal{C}$ is* ***proper*** *if it contains no complex planes, i.e if x, y are independent vectors in E then $Span(x, y) \not\subset \mathcal{C}$.*

A proper, closed, complex cone $\mathcal{C}$ will be refered to as a $\mathbb{C}$-cone.

We now let $\hat{\mathbb{C}}$ be the Riemann sphere, and define, for any pair $(x, y) \in \mathcal{C}^*$

$$D(x, y) := \{\lambda \in \hat{\mathbb{C}} : (1 + \lambda)x + (1 - \lambda)y \in \mathcal{C}\} \subset \hat{\mathbb{C}}$$

where $\infty \in D(x, y)$ if and only if $x - y \in \mathcal{C}$. We denote the interior (for the spherical topology on $\hat{\mathbb{C}}$) of this "slice" by $D^0(x, y)$.

Recall that the standard **hyperbolic metric** on the complex unit disk $\mathbb{D}$ is defined by

$$d_{\mathbb{D}}(x, y) = \inf_{\substack{\gamma:[0,1]\to\mathbb{D},\ C^1 \\ (\gamma(0),\gamma(1))=(x,y)}} \int_\gamma \frac{2|dz|}{1 - |z|^2} = \log\left(\frac{|1 - x\bar{y}| + |x - y|}{|1 - x\bar{y}| - |x - y|}\right)$$

For any open connected subset $U \subset \hat{\mathbb{C}}$ avoiding at least 3 points, we define a **hyperbolic metric** on U by transporting the hyperbolic metric from $\mathbb{D}$:

$$d_U(x, y) = d_{\mathbb{D}}(\phi^{-1}(x), \phi^{-1}(y))$$

where $\phi : \mathbb{D} \to U$ is a bi-conformal map given by Riemann's conformal mapping theorem. We are now ready to define the **complex gauge**

Definition A.5
Given a $\mathbb{C}$-cone $\mathcal{C}$, we define the ***complex gauge*** *$d_{\mathcal{C}} : \mathcal{C}^* \times \mathcal{C}^* \to [0, +\infty]$ as follows:*

- *If $x, y \in \mathcal{C}^*$ are co-linear, we set $d_{\mathcal{C}}(x, y) = 0$.*
- *If $x, y \in \mathcal{C}^*$ are linearly independent, and that -1 and 1 belong to the same connected component U of $D^0(x, y)$, then we set*

$$d_{\mathcal{C}}(x, y) = d_U(-1, 1) > 0 \tag{A.2.1}$$

- *In all remaining cases, we set $d_{\mathcal{C}}(x, y) = \infty$.*

For $\mathcal{C}' \subset \mathcal{C}$ a sub-cone of the $\mathbb{C}$-cone $\mathcal{C}$, one defines

$$diam_{\mathcal{C}}(\mathcal{C}') := \sup_{x,y\in\mathcal{C}'} d_{\mathcal{C}}(x, y) \in [0, +\infty] \tag{A.2.2}$$

Note that the previously defined complex gauge $d_{\mathcal{C}}$ is not necessarily a metric as it does not separates co-linear vectors, nor necessarily verifies the triangular inequality !
However, we will see that the complex gauge $d_{\mathcal{C}}$ is, in a sense, "similar enough" to a metric for our purposes, particularly that it is symmetric and projective:

Lemma A.4
Let $\mathcal{C}$ be a $\mathbb{C}$-cone, and let $x, y \in \mathcal{C}^*$, and $a \in \mathbb{C}^*$. Then one has

$$d_{\mathcal{C}}(x, y) = d_{\mathcal{C}}(y, x) = d_{\mathcal{C}}(ax, y) = d_{\mathcal{C}}(x, ay)$$

Lemma A.5
Let $\mathcal{L} : E_1 \to E_2$ be a complex linear map between topological vector spaces, and let $\mathcal{C}_1 \subset E_1$, $\mathcal{C}_2 \subset E_2$ be $\mathbb{C}$-cone such that $\mathcal{L}(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$.
Then the map $\mathcal{L} : (\mathcal{C}_1^*, d_{C_1}) \to (\mathcal{C}_2^*, d_{C_2})$ is a contraction.

Furthermore, if the image has a finite diameter (i.e $\Delta = diam_{\mathcal{C}_2^*}\mathcal{L}(C_1^*) < \infty$), then the contraction is both strict and uniform. More precisely, there exists $\eta < 1$ depending only on Δ, so that

$$\forall x, y \in C_1^*, d_{\mathcal{C}_2}(\mathcal{L}(x), \mathcal{L}(y)) < \eta d_{\mathcal{C}_1}(x, y)$$

Proof. Let $x, y \in \mathcal{C}_1^*$, and set $D_1 = D(x, y, C_1)$ and $D_2 = D(\mathcal{L}(x), \mathcal{L}(y), \mathcal{C}_2)$, so that one has:

$$\{-1, 1\} \subset D_1 \subset D_2 \subset \hat{\mathbb{C}}$$

We can always assume that $\mathcal{L}(x), \mathcal{L}(y)$ are linearly independent and that D_1, D_2 are hyperbolic (otherwise $d_{C_2}(\mathcal{L}(x), \mathcal{L}(y)) = 0$ and we are done). Using the classical fact that decreasing a domain increases hyperbolic distance, one gets

$$d_{C_2}(\mathcal{L}(x), \mathcal{L}(y)) \leq d_{C_1}(x, y)$$

i.e $\mathcal{L} : (\mathcal{C}_1^*, d_{C_1}) \to (\mathcal{C}_2^*, d_{C_2})$ is a (weak) contraction.

Now we assume that $\Delta < \infty$. Then $-1, 1$ belong to the same connected component, V of D_2^0, and we can assume without loss of generality that $-1, 1$ belong to the same connected component U of D_1^0 (else $d_{C_1}(x, y) = \infty$ and we are done). It follows from our assumptions that $U \subsetneq V$, and that $diam_V(U) < \infty$.
One can thus choose a $q \in U$ and a $p \in V \backslash U$ for which $d_V(q, p) \leq \Delta$. Note that the inclusion $U \hookrightarrow V\backslash\{p\}$ as non expanding (for the hyperbolic metric) and that the inclusion $V\backslash\{p\} \hookrightarrow V$ is a contraction whose hyperbolic derivative is strictly smaller that some $\eta = \eta(\Delta) < 1$ on $B_V(p, \Delta)^*$ the punctured Δ neighborhood of p. Therefore, the composed inclusion map $U \hookleftarrow V\backslash\{p\} \hookrightarrow V$ has hyperbolic derivative smaller than $\eta(\Delta)$ at $\lambda \in B_V(p, \Delta)^*$. One may take λ lying on a geodesic joining -1 and 1 in U, so that one gets

$$d_{\mathcal{C}_2}(\mathcal{L}(x), \mathcal{L}(y)) = d_V(-1, 1) \leq \eta d_U(-1, 1) = \eta d_{\mathcal{C}_1}(x, y)$$

□

The notion of **regularity** for Birkhoff cones, defined in def.A.2 can be transposed verbatim to the setting of $\mathbb{C}$-cones. Thus, we also get a way to compare the complex gauge $d_{\mathcal{C}}$ and the norm on the ambient Banach space:

Lemma A.6
Let $\mathcal{C}$ be a $\mathbb{C}$-cone of K-bounded sectional aperture. If $x, y \in \mathcal{C}^*$, and if $\ell = \ell_{x,y}$ is the (outer) regularity functional associated to the sub-cone $Span(x,y) \cap \mathcal{C}$, then one has:

$$\left\| \frac{x}{\langle \ell, x\rangle} - \frac{y}{\langle \ell, y\rangle} \right\| \leq \frac{4K}{\|\ell\|} \tanh\left(\frac{d_{\mathcal{C}}(x,y)}{4}\right) \leq K \frac{d_{\mathcal{C}}(x,y)}{4} \tag{A.2.3}$$

Proof. Taking the regularity functional $\ell = \ell_{x,y}$ so that $\|\ell\| = K$, one can write for every $u \in Span(x,y)$:

$$\|u\| \leq |\langle \ell, u\rangle| \leq K.\|u\|$$

Consider $x_1 = \frac{x}{\langle \ell, x\rangle}$ and $y_1 = \frac{y}{\langle \ell, y\rangle}$, and let $u_\lambda = (1+\lambda)x_1 + (1-\lambda)y_1$ for any $\lambda \in \mathbb{C}$. When $u_\lambda \in \mathcal{C}$, one has by regularity

$$|u_\lambda| \leq |\langle \ell, u_\lambda\rangle| = 2 \tag{A.2.4}$$
$$|\lambda|.\|x_1 - y_1\| \leq |u_\lambda|.\|x_1 - y_1\| \leq 4 \tag{A.2.5}$$

Setting $R = \frac{4}{\|x_1 - y_1\|} \in [2, \infty]$, one sees that $D(x_1, y_1) \subset B(0, R)$. As expanding the domain decreases the hyperbolic distance, one has:

$$d_{\mathcal{C}}(x,y) = d_{D^0(x_1,y_1)}(-1,1) \geq d_{B(0,R)}(-1,1) = d_{\mathbb{D}}(-1/R, 1/R) = 2\log\left(\frac{1+R}{1-R}\right)$$

Taking the inverse, one gets the announced bound:

$$\frac{\|x_1 - y_1\|}{4} = \frac{1}{R} \leq \tanh\left(d_{\mathcal{C}}(x,y)/4\right) \leq d_{\mathcal{C}}(x,y)/4$$

□

Lemma A.7
Let $\mathcal{C}$ be a $\mathbb{C}$ cone with K-bounded sectional aperture, and let $x \in \mathcal{C}^*$, $y \in E$. Suppose that there exists a $r > 0$, such that $x + ty \in \mathcal{C}^*$ for every $t \in \mathbb{C}$, $|t| < r$. Then one has

$$d_{\mathcal{C}}(x, x+sy) \leq \frac{2|s|}{r} + \underset{s\to 0}{o}(|s|) \tag{A.2.6}$$
$$\|y\| \leq \frac{K}{r}\|x\| \tag{A.2.7}$$

Proof. Let $|s| < r$. Then one has

$$D(x, x+sy) = \{\lambda \in \hat{\mathbb{C}},\ (1+\lambda)x + (1-\lambda)(x+sy) \in \mathcal{C}\} \tag{A.2.8}$$
$$= \{\lambda \in \hat{\mathbb{C}},\ x + \frac{1-\lambda}{2}sy \in \mathcal{C}\} \tag{A.2.9}$$

by using that the cone is $\mathbb{C}$ invariant. Our assumptions imply that $D(1, \frac{2r}{|s|}) \subset D(x, x+sy)$, and as shrinking the domain increases hyperbolic distance, one has

$$d_{\mathcal{C}}(x, x+sy) \leq d_{D(1,\frac{2r}{|s|})}(-1, 1) = d_{\mathbb{D}}(0, |s|/r) = \log\left(\frac{1+|s|/r}{1-|s|/r}\right) = \frac{2|s|}{r} + o(|s|)$$

as announced.

For the second item, take the regularity functional $\ell_{x,y} = \ell$ associated with the sub-cone $Span(x, y) \cap \mathcal{C}$, normalized so that $\|\ell\| = K$.
By our assumption, for every $t \in \mathbb{C}$, $|t| < r$, one has $0 < |\langle \ell, x+ty\rangle| = |\langle \ell, x\rangle + t\langle \ell, y\rangle|$, which implies $r|\langle \ell, y\rangle| \leq |\langle \ell, x\rangle|$.
Up to multiplying x, y by some appropriate complex phase, we can assume $\langle \ell, x\rangle \geq \langle \ell, ry\rangle > 0$.
Thus we get

$$2r\|y\| \leq \|x+ry\| + \|x-ry\| \leq \langle \ell, x+ry\rangle + \langle \ell, x-ry\rangle = 2\langle \ell, x\rangle \leq 2K\|x\|$$

which proves the claim.

The estimates in this lemma are key to obtain regularity and invertibility for the projection onto the affine hyperplane $\{\ell = 1\}$ of a regular $\mathbb{C}$-cone $\mathcal{C}$ with regularity functional ℓ. □
We now turn to refinements of our contraction principle A.5:

Theorem A.4
Let E be a complex Banach space, $\mathcal{C} \subset E$ be a $\mathbb{C}$-cone contraction with K-bounded sectional aperture and let $\mathcal{L} \in L(E)$ be a strict cone contraction (i.e $\mathcal{L} : \mathcal{C}^* \to \mathcal{C}^*$ and $\Delta = diam_{\mathcal{C}^*}\mathcal{L}(\mathcal{C}^*) < \infty$). Let $\eta < 1$ be as in theorem A.5. Then:

- *$\mathcal{C}$ contains a unique $\mathcal{L}$-invariant complex line, $\mathbb{C}h$.*
  We define $\lambda \in \mathbb{C}^*$ by $\mathcal{L}(h) = \lambda.h$. Then
- *There exists constants $R, C < \infty$ and a map $c : \mathcal{C} \to \mathbb{C}$ such that for every $x \in \mathcal{C}$ and $n \geq 1$, one has*

$$\|\lambda^{-n}\mathcal{L}^n x - c(x)h\| \leq C\eta^{n-1}\|x\| \tag{A.2.10}$$

$$\|c(x)h\| \leq R\|x\| \tag{A.2.11}$$

Proof. We are going to construct the $\mathcal{L}$ invariant line generator $h \in \mathcal{C}^*$ recursively, with a Picard iteration scheme.
Let $e_0 \in \mathcal{C}^*$, and define $e_1 = \dfrac{\mathcal{L}(e_0)}{\|\mathcal{L}(e_0)\|}$. Given $n \geq 1$ and assuming e_n is constructed, we choose a regularity functional $\ell_{e_n,\mathcal{L}e_n} = \ell_n$ associated to the sub-cone $Span(e_n, \mathcal{L}e_n) \cap \mathcal{C}$, and normalized so that $\|\ell_n\| = K$. We set $\lambda_n = \dfrac{\langle \ell_n, \mathcal{L}e_n\rangle}{\langle \ell_n, e_n\rangle}$, and note that $0 < |\lambda_n| \leq K\|\mathcal{L}\|$.
The next element of the sequence $(e_n)_{n\geq 0}$ is

$$e_{n+1} = \frac{\lambda_n^{-1}\mathcal{L}e_n}{\|\lambda_n^{-1}\mathcal{L}e_n\|} \tag{A.2.12}$$

By (A.2.3) and an iterated use of theorem A.5, one has:

$$\|\frac{e_n}{\langle \ell_n, e_n\rangle} - \frac{\mathcal{L}e_n}{\langle \ell, \mathcal{L}e_n\rangle}\| \leq d_{\mathcal{C}}(e_n, \mathcal{L}e_n) \leq diam_{\mathcal{C}}\mathcal{L}^n(\mathcal{C}^*) \leq \Delta\eta^{n-1}. \tag{A.2.13}$$

Multiplying this last inequality by $|\langle \ell_n, e_n\rangle|$ (resp. $|\langle \ell_n, \mathcal{L}e_n\rangle|$), and using $|\langle \ell_n, e_n\rangle| \le K\|e_n\| = K$ (resp $|\langle \ell_n, \mathcal{L}e_n\rangle| \le K\|\mathcal{L}\|$), one obtains

$$\|e_n - \lambda_n^{-1}\mathcal{L}e_n\| \le K\Delta\eta^{n-1} \tag{A.2.14}$$
$$\|\lambda_n e_n - \mathcal{L}e_n\| \le \|\mathcal{L}\|K\Delta\eta^{n-1} \tag{A.2.15}$$

This first inequality, together with $\|e_n\| = 1$ and continuity of $x \mapsto \dfrac{x}{\|x\|}$ on $E\backslash\{0\}$ gives

$$\|e_n - e_{n+1}\| \le 2K\Delta\eta^{n-1} \tag{A.2.16}$$

showing that the sequence $(e_n)_{n\ge 0}$ is Cauchy, and thus converges towards $h = \lim_{n\to\infty} e_n$. We have $h \in \mathcal{C}^*$ because the cone $\mathcal{C}$ was assumed to be closed.

Now we write

$$(\lambda_{n+1} - \lambda_n)e_{n+1} = (\mathcal{L} - \lambda_n)e_n + (\lambda_{n+1} - \mathcal{L})e_{n+1} + (\mathcal{L} - \lambda_n)(e_{n+1} - e_n)$$

Using (A.2.14) and (A.2.16), one gets

$$|\lambda_n - \lambda_{n+1}| \le \|(\lambda_n - \lambda_{n+1})e_{n+1}\| \tag{A.2.17}$$
$$\le \|(\mathcal{L} - \lambda_n)e_n\| + \|(\mathcal{L} - \lambda_{n+1})e_{n+1}\| + \|(\mathcal{L} - \lambda_n)\|.\|e_{n+1} - e_n\| \tag{A.2.18}$$
$$\le K\|\mathcal{L}\|\Delta\eta^{n-1}(1 + \eta + 2 + 2K) \tag{A.2.19}$$

which establish the convergence $\lambda_n \to \lambda$. But now note that $\|\mathcal{L}(h) - \lambda.h\| = \lim_n \|\mathcal{L}e_n - \lambda_n e_n\| = 0$, and thus $\mathcal{L}h = \lambda h \in \mathcal{C}^*$.
Therefore $\lambda \ne 0$ and $\mathbb{C}h \subset \mathcal{C}$ is a $\mathcal{L}$-invariant line.

For uniqueness, assume the existence of $k \in \mathcal{C}^*$ such that $\mathbb{C}k \subset \mathcal{C}$ is a $\mathcal{L}$-invariant complex line. Then

$$d_{\mathcal{C}}(h,k) = d_{\mathcal{C}}(\mathcal{L}^n h, \mathcal{L}^n k) \le \eta^{n-1}\Delta < \infty$$

for any $n \ge 1$, and thus $d_{\mathcal{C}}(h,k) = 0$, i.e $\mathbb{C}h = \mathbb{C}k$.

We now turn to estimates (A.2.10). Let $x \in \mathcal{C}^*$, and define for $n \ge 1$, $x_n = \mathcal{L}^n x$. We now take $m_n \in E'$ the regularity functional associated to the sub-cone $Span(x_n, h) \cap \mathcal{C}$, and set $c_n = \frac{\langle m_n, \lambda^{-n}x_n\rangle}{\langle m_n, h\rangle}$, for which we have $0 < |c_n| < K\|\lambda^{-n}x_n\|$.
Using again lemma A.6, one gets

$$\left\|\frac{x_n}{\langle m_n, x_n\rangle} - \frac{h}{\langle m_n, h\rangle}\right\| \le \frac{K}{\|m_n\|}d_{\mathcal{C}}(\mathcal{L}^n x, \mathcal{L}^n h) \le \frac{K}{\|m_n\|}diam_{\mathcal{C}}(\mathcal{L}^n\mathcal{C}^*) \le \frac{K}{\|m_n\|}\Delta\eta^{n-1} \tag{A.2.20}$$

This last inequality, written for $\lambda^{-n}x_n$ and multiplied by $|\langle m_n, \lambda^{-n}x_n\rangle|$, becomes

$$\|\lambda^{-n}x_n - c_n h\| \le K\|\lambda^{-n}x_n\|\Delta\eta^{n-1} \tag{A.2.21}$$

Furthermore,

$$\|\lambda^{-n-1}x_{n+1} - \lambda^{-n}x_n\| = \|\lambda^{-1}\mathcal{L}[\lambda^{-n}x_n - c_n h] + c_n h - \lambda^{-n}x_n\| \tag{A.2.22}$$
$$\le \left[1 + \|\lambda^{-1}\mathcal{L}\|\right] K\|\lambda^{-n}x_n\|\Delta\eta^{n-1} \tag{A.2.23}$$

This last bound yields

$$\|\lambda^{-n-1}x_{n+1}\| \leq \left[1 + \left[1 + \|\lambda^{-1}\mathcal{L}\|\right] K\|\lambda^{-n}x_n\|\Delta\eta^{n-1}\right] \|\lambda^{-n}x_n\|$$

so we get the following uniform bound:

$$\|\lambda^{-n}x_n\| \leq \prod_{k\geq 0} \left[1 + \left[1 + \|\lambda^{-1}\mathcal{L}\|\right] K\Delta\eta^k\right] \|\lambda^{-1}x_1\| \tag{A.2.24}$$

$$\leq \exp(1 + \|\lambda^{-1}\mathcal{L}\|\frac{K\Delta}{1-\eta})\|\lambda^{-1}\mathcal{L}\|x\| := R\|x\| \tag{A.2.25}$$

Now we write

$$(c_{n+1} - c_n)h = c_{n+1}h - \lambda^{-n-1}x_{n+1} + \lambda^{-1}\mathcal{L}\left[\lambda^{-n}x_n - c_n h\right]$$

so that one has

$$|c_{n+1} - c_n| \leq \left(\eta + \|\lambda^{-1}\mathcal{L}\|\right) K\Delta\eta^{n-1}R\|x\|$$

Thus $c^* = \lim_{n\to\infty} c_n$ exists (the sequence is Cauchy) and using the rest in the above convergent series, one gets the following bound:

$$|c^* - c_n| \leq \eta^{n-1}\frac{KR\Delta\|x\|\left[\eta + \|\lambda^{-1}\mathcal{L}\|\right]}{1-\eta}$$

Using this last bound together with (A.2.21) and (A.2.24), one gets

$$|\lambda^{-n}x_n - c^*h| \leq \frac{1 + \|\lambda^{-1}\mathcal{L}\|}{1-\eta}K\Delta\eta^{n-1}R\|x\| \leq C\eta^{n-1}\|x\|$$

which implies that $c^* = c(x)$ only depends on the choice of x, not on the choice of m_n. We also get the uniform bound

$$|c(x)| = \|c(x)h\| = \lim_n \|\lambda^{-n}x_n\| \leq R\|x\| \tag{A.2.26}$$

□ We can now turn to our ultimate refinement for the contraction principle: under an additional technical requirement for the cone $\mathcal{C}$ (namely the *reproducing* condition), the map $x \in E \mapsto c(x) \in \mathbb{C}$ is linear (which is bounded by virtue of (A.2.26)), i.e $\mathcal{L}$ admits a spectral gap.

We start by defining what it means to be reproducing for a $\mathbb{C}$-cone (note the analogy with definition A.2).

Definition A.6

Let E be a complex Banach space, and let $\mathcal{C} \subset E$ be a $\mathbb{C}$-cone. $\mathcal{C}$ is said to be ***reproducing*** *if there exists a (real) constant $g < +\infty$, such that every $x \in E$ decomposes into $x = x_1 + x_2$ and*

$$\|x_1\| + \|x_2\| \leq g\|x\| \tag{A.2.27}$$

Theorem A.5

Let $\mathcal{L} \in L(E)$ and $\mathcal{C} \subset E$ be a $\mathbb{C}$-cone of K-bounded sectional aperture and reproducing. Suppose that $\mathcal{L} : \mathcal{C}^* \to \mathcal{C}^*$ is a strict cone contraction, with $\Delta := diam_{\mathcal{C}}\mathcal{L}\mathcal{C}^* < \infty$. Then $\mathcal{L}$ has a spectral gap.

Proof. Let $x \in E$, and take g be the reproducing constant from definition A.6. Taking $x_1, x_2 \in \mathcal{C}$ so that $x = x_1 + x_2$, and $\|x_1\| + \|x_2\| \leq g\|x\|$. Applying theorem A.2.10 to x_1 and x_2 yields the existence of $c^* := c(x_1) + c(x_2)$, which satisfies $|c^*| \leq gR\|x\|$. Furthermore,

$$\|\lambda^{-n}\mathcal{L}^n x - c^* h\| \leq \|\lambda^{-n}\mathcal{L}^n x_1 - c(x_1)h\| + \|\lambda^{-n}\mathcal{L}^n x_2 - c(x_2)h\| \leq C\eta^{n-1}\left[\|x_1\| + \|x_2\|\right] \leq Cg\eta^{n-1}\|x\|$$

so that one has $c^* = c(x) := \lim_n \lambda^{-n}\mathcal{L}^n x = c(x_1) + c(x_2)$ depends only on x and nor on the choice of decomposition $x = x_1 + x_2$.
By linearity of $\mathcal{L}$, $x \in E \mapsto c(x) \in \mathbb{C}$ is a linear form, bounded by virtue of (A.2.26) with norm smaller than gR. From now on, we will denote $\langle \nu, x \rangle := c(x) \in \mathbb{C}$.
We have shown that for every $x \in E$, every $n \geq 1$,

$$\|\lambda^{-n}\mathcal{L}^n x - h.\langle \nu, x \rangle\| \leq gC\eta^{n-1}\|x\| \tag{A.2.28}$$

Thus $\mathcal{L}$ admits a spectral gap, with $\lambda \in \mathbb{C}^*$ its maximal, simple eigenvalue, $x \mapsto \langle \nu, x \rangle .h$ the associated eigenprojector, and the remainder of its spectrum contained in a disk of radius smaller than $\eta|\lambda|$. □

We now present a fundamental example of contracted cone:

Proposition A.1
For E a complex Banach space, we let $h \in E$, $\nu \in E'$ such that $\langle \nu, h \rangle = 1$. Let P be the associated one dimensional projector, i.e $Px := \langle \nu, x \rangle .h$. For $\sigma \in (0, +\infty)$, we define the $\mathbb{C}$-cone

$$\mathcal{C}_\sigma := \{x \in E : \ \|x - Px\| \leq \sigma\|Px\|\} \tag{A.2.29}$$

This family of $\mathbb{C}$-cones has the following properties:

- *For any $\sigma \in (0, \infty)$, $\mathcal{C}_\sigma$ is inner regular. More precisely, $B_E(h, \dfrac{\sigma\|h\|}{\|1 - P\| + \sigma\|P\|}) \subset \mathcal{C}$.*
- *$K(\mathcal{C}_\sigma) \leq (1+\sigma)\|P\|$, for any $\sigma \in (0, +\infty)$, i.e $\mathcal{C}_\sigma$ is outer regular (or of bounded aperture).*
- *Let $0 < \sigma' < \sigma < +\infty$. Then trivially $\mathcal{C}_{\sigma'} \subset \mathcal{C}_\sigma$, and*

$$diam_{\mathcal{C}_\sigma}\mathcal{C}^*_{\sigma'} < +\infty$$

Proof.

- For the first item, we proceed in 2 times: first we show that $B_E(h, r_\sigma) \subset \mathcal{C}_\sigma$, where $r_\sigma := \dfrac{\sigma\|h\|}{\|1 - P\| + \sigma\|P\|}$, and then that any ball $B_E(h, r) \subset \mathcal{C}_\sigma$ satisfies $r \leq r_\sigma$.

  Let $x \in B_E(h, r_\sigma)$. Then one has that $Px \in B_E(h, \|P\|r_\sigma)$, so that $\|Px\| \geq \|h\| - \|P\|r_\sigma$. Therefore,

$$\frac{\|(1-P)x\|}{\|Px\|} \leq \frac{\|1-P\|r_\sigma}{\|h\| - \|P\|r_\sigma} \tag{A.2.30}$$

$$= \frac{\|1-P\|\sigma\|h\|}{\|1-P\| + \sigma\|P\|} \cdot \frac{\|1-P\| + \sigma\|P\|}{\|1-P\|.\|h\|} \tag{A.2.31}$$

$$= \sigma \tag{A.2.32}$$

where we used in the first line $(1-P)x = (1-P)(x-h)$.

Let $r > 0$ be such that $B_E(h,r) \subset \mathcal{C}_\sigma$. Then one can write, as before

$$\frac{\|(1-P)x\|}{\|Px\|} \leq \frac{\|(1-P)\|r}{\|h\| - \|P\|r} \tag{A.2.33}$$

the right-hand term satisfies $\frac{\|(1-P)\|r}{\|h\|-\|P\|r} \leq \sigma$ if and only if

$$\|(1-P)\|r \leq \sigma\|h\| - \sigma\|P\|r \tag{A.2.34}$$

$$r\left[\|1-P\| + \sigma\|P\|\right] \leq \sigma\|h\| \tag{A.2.35}$$

$$r \leq \frac{\sigma\|h\|}{\|1-P\| + \sigma\|P\|} = r_\sigma \tag{A.2.36}$$

establishing the claim.

- For the second item, we start by remarking that for all $x \in E$, $\|Px\| = |\langle \nu, x\rangle|.\|h\|$, so that clearly $\|P\| = \|\nu\|.\|h\|$.
  From the definition of $\mathcal{C}_\sigma$, it follows immediately that

  $$\|x\| \leq \|Px\| + \|(1-P)x\| \leq (1+\sigma)\|Px\| = (1+\sigma)|\langle \nu, x\rangle|.\|h\|$$

  Multiplying by $\|\nu\|$, one gets $\|\nu\|.\|x\| \leq (1+\sigma)|\langle \nu, x\rangle|.\|P\|$, so that by definition of the aperture $K(\mathcal{C}_\sigma)$ of $\mathcal{C}_\sigma$,

  $$K(\mathcal{C}_\sigma) \leq (1+\sigma)\|P\|$$

  It follows from the first 2 items that $\mathcal{C}_\sigma$ is a **regular** $\mathbb{C}$-cone: in particular, it has (uniformly) bounded sectional aperture and it is reproducing.

- For the last statement, we refer to [59, §3] or to [71, Example 3.9] □

This family $(\mathcal{C}_\sigma)_{\sigma>0}$ of regular $\mathbb{C}$-cone has a particularly nice property: in a sense, it *characterizes* the spectral gap presence:

Theorem A.6
Let E be a complex Banach space, and let $\mathcal{L} : E \to E$ be a bounded operator. Then $\mathcal{L}$ admits a spectral gap if and only if it contracts a regular $\mathbb{C}$-cone.

Proof. The reverse implication is the content of theorem A.5.

We now turn to the direct implication: suppose that $\mathcal{L}$ admits a spectral gap, with dominating eigenvalue λ, associated eigenvectors $h \in E$ and $\nu \in E'$. One can construct the one dimensional spectral projector $Px := h\langle \nu, x\rangle$, and given $\theta \in (\eta|\lambda|, |\lambda|)$, we define $\|.\|_\theta$ by

$$\|x\|_\theta := \|Px\| + \sum_{k\geq 0} \theta^- k\|\mathcal{L}^k[1-P]x\|$$

Our choice of θ insures both convergence of the above sum and the equivalence of $\|.\|_\theta$ and $\|.\|_E$. We fix such a θ, and now we can define the family of regular $\mathbb{C}$-cones $(\mathcal{C}_{\sigma,\theta})_{\sigma>0}$ by (A.2.29) with $\|.\|_\theta$ instead of $\|.\|_E$.

Let us now show that given a $\sigma > 0$, $\mathcal{L} : \mathcal{C}_\sigma^* \to \mathcal{C}_\sigma^*$ is a strict and uniform contraction: in that endeavor, one needs to estimate $\|(1-P)\mathcal{L}x\|_\theta$. But it is easy to see that

$$\|(1-P)\mathcal{L}x\|_\theta = \sum_{k\geq 0} \theta^{-k}\|\mathcal{L}^{k+1}[1-P]x\| \tag{A.2.37}$$

$$= \theta \sum_{k\geq 1} \theta^{-k}\|\mathcal{L}^{k}[1-P]x\| \tag{A.2.38}$$

$$\leq \theta\|(1-P)x\|_\theta \tag{A.2.39}$$

$$\leq \frac{\theta\sigma}{|\lambda|}\|P\mathcal{L}x\|_\theta \tag{A.2.40}$$

where we used $(1-P)^2 = 1-P$, $\|(1-P)x\|_\theta = \sum_{k\geq 0}\theta^{-k}\|\mathcal{L}^k[1-P]x\|$ and that $\mathcal{L}P = P\mathcal{L} = \lambda.P$. Therefore, $\mathcal{L}(\mathcal{C}_{\sigma,\theta}^*) \subset \mathcal{C}_{\sigma',\theta}^*$ with $\sigma' := \frac{\theta}{|\lambda|}\sigma < \sigma$, which shows that it is indeed a strict and uniform cone contraction. □

A.2.1 Canonical complexification of a Birkhoff cone

Now that we saw how a complex cone generalizes the notion of a Birkhoff cone, one can wonder how to construct a complex cone from a Birkhoff cone. It turns out that there is a canonical way to do so, in an isometric way (with respect to the projective metric/complex gauge).

Definition A.7

- *Let $(X, \|.\|)$ be a real Banach space. We define its **canonical complexification** $X_\mathbb{C}$ by $X_\mathbb{C} := X \oplus iX$. When endowed with the norm $\|.\|_\mathbb{C}$, defined by*

$$\|x+iy\|_\mathbb{C} := \sup_{\substack{\ell\in X' \\ \|\ell\|_{X'}\leq 1}} \{|\langle \ell, x\rangle + i\langle \ell, y\rangle|\}$$

  it is a complex Banach space.

- *Let $\mathcal{C} \subset X$ be a Birkhoff cone. We define its canonical complexification $\mathcal{C}_\mathbb{C}$ by*

$$\mathcal{C}_\mathbb{C} := \{u \in X_\mathbb{C}, \Re(\langle m, u\rangle\overline{\langle \ell, u\rangle}) \geq 0,\ \forall m, \ell \in \mathcal{C}'\} \tag{A.2.41}$$

  Equivalently, it can be defined by $\mathcal{C}_\mathbb{C} := \mathbb{C}^*(\mathcal{C} + i\mathcal{C})$.

Which geometric properties of the original Birkhoff cone remain after going through complexification ? This natural question is answered in the next proposition:

Proposition A.2 ([71], Prop 5.4)
Let $\mathcal{C} \subset X$ be a Birkhoff cone. If $\mathcal{C}$ is inner regular (respectively generating, outer regular, of bounded sectional aperture), then so is its canonical complexification $\mathcal{C}_\mathbb{C}$.

One of the central tool of the theory of Birkhoff cones is the Hilbert projective metric $d_\mathcal{C}$ (see definition A.3), which allows one to formulate a contraction principle theorem A.1. The **complex gauge** of Dubois [21] $\delta_\mathcal{C}$ is the analogue of the Hilbert metric in the setting of complex cones, in the following sense:

Theorem A.7 ([22], theorem 4.3)
Let $\mathcal{C}$ be a Birkhoff cone, and let $\mathcal{C}_\mathbb{C}$ denote its canonical complexification (A.2.41). Then one has

- *The natural inclusion* $(\mathcal{C}^*, d_{\mathcal{C}}) \hookrightarrow (\mathcal{C}_{\mathbb{C}}, \delta_{\mathcal{C}})$ *is an isometric embedding.*
- *Let* $\mathcal{L}$ *be a bounded operator mapping* $\mathcal{C}^*$ *to itself. Then the complexification of* $\mathcal{L}$ *(still denoted by* $\mathcal{L}$*) maps* $\mathcal{C}^*_{\mathbb{C}}$ *to itself.*
  Furthermore, if $\Delta_{\mathbb{R}} := diam_{\mathcal{C}}(\mathcal{L}(\mathcal{C})^*)$ *and* $\Delta_{\mathbb{C}} := \sup_{x,y\in\mathcal{C}_{\mathbb{C}}} \delta_{\mathcal{C}}(\mathcal{L}x, \mathcal{L}y)$*, then*

$$\Delta_{\mathbb{C}} \leq 3\Delta_{\mathbb{R}} \tag{A.2.42}$$

- *Assume there exists a* $\rho > 0$*, such that* $B(\mathcal{L}\phi, \rho\|\mathcal{L}\phi\|) \subset \mathcal{C}$*, for every* $\phi \in \mathcal{C}^*$*. Then*

$$B_{\mathbb{C}}(\mathcal{L}\phi, \frac{\rho}{2}\|\mathcal{L}\phi\|_{\mathbb{C}}) \subset \mathcal{C}_{\mathbb{C}}$$

Proof of theorem A.7 The first 2 statements are taken from [22, theorem 4.3], and we refer to this paper for a proof.

For the last point, we start by noting that if $B(x, r) \subset \mathcal{C}$ for some $x \in \mathcal{C}$, then $B_{\mathbb{C}}(x, \frac{r}{2}) \subset \mathcal{C}_{\mathbb{C}}$. If $\phi \in \mathcal{C}^*_{\mathbb{C}}$, we write $\phi = \lambda(\phi_1 + \phi_2)$, with $(\phi_1, \phi_2) \in \mathcal{C}^2$, and $\lambda \in \mathbb{C}^*$, then

$$B_{\mathbb{C}}(\mathcal{L}\phi, \frac{\rho}{2}\|\mathcal{L}\phi\|_{\mathbb{C}}) \subset \lambda.B_{\mathbb{C}}(\mathcal{L}\phi_1, \frac{\rho}{2}\|\mathcal{L}\phi_1\|) + \lambda.B_{\mathbb{C}}(\mathcal{L}\phi_2, \frac{\rho}{2}\|\mathcal{L}\phi_2\|) \subset \mathbb{C}^*(\mathcal{C} + i\mathcal{C}) = \mathcal{C}_{\mathbb{C}} \tag{A.2.43}$$

which establishes the claim. □

A.3 Random product of cone contractions

In this section, E will denote indifferently a real or complex Banach space.

Definition A.8
Let E *be a Banach space, let* $\mathcal{C} \subset E$ *be a regular* $\mathbb{R}$ *cone, and let* $\mathcal{L} : E \to E$*, be a bounded operator such that:*

1. $\mathcal{L}(\mathcal{C}^*) \subsetneq \mathcal{C}^*$
2. $diam(\mathcal{L}(\mathcal{C}^*)) \leq \Delta$
3. *There exists* $\rho > 0$ *such that for every* $\phi \in \mathcal{C}^*$*,* $B_E(\mathcal{L}\phi, \rho\|\mathcal{L}\phi\|) \subset \mathcal{C}$*.*

We will denote by $\mathcal{M}_{\mathcal{C}}(\Delta, \rho)$ *(or just* $\mathcal{M}(\Delta, \rho)$*) the set of all cone contractions subject to this uniform bound.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\tau : \Omega \to \Omega$ be an invertible and ergodic map. Let E be a Banach space, and let $(\mathcal{L}_\omega)_{\omega\in\Omega}$ be a family of bounded operators on E such that the map $\boldsymbol{\mathcal{L}} :\ \omega \in \Omega \mapsto \mathcal{L}_\omega \in \mathcal{M}(\Delta, \rho)$ is measurable and $\mathbb{P}$-essentially bounded.
In other terms, we choose randomly (with probability distribution $\mathbb{P}$) a strict and uniform contraction of the regular $\mathbb{R}$-cone $\mathcal{C}$. Then one has

Theorem A.8
Let $\mathcal{C}$ *be a proper, closed, convex and regular cone.*
Let $\ell \in E'$ *be the norm one linear form given by the outer regularity of* $\mathcal{C}$*, i.e such that there is* $K > 0$ *such that for every* $x \in \mathcal{C}^*$*,* $\langle \ell, x\rangle \geq \frac{\|x\|}{K}$*.*

Let $\boldsymbol{\mathcal{L}} \in L^\infty(\Omega, \mathcal{M}(\Delta, \rho))$. We define the associated projection on $\mathcal{C} \cap \{\ell = 1\}$, $\pi_\omega : \mathcal{C} \cap \{\ell = 1\} \to \mathcal{C} \cap \{\ell = 1\}$, by

$$\pi_\omega(x) := \frac{\mathcal{L}_\omega x}{\langle \ell, \mathcal{L}_\omega x\rangle} \tag{A.3.1}$$

for $x \in \mathcal{C}^*$.

This induces an operator on $L^\infty(\Omega, \mathcal{C}^*)$, $\boldsymbol{\pi} :\ L^\infty(\Omega, \mathcal{C}) \to L^\infty(\Omega, \mathcal{C} \cap \{\ell = 1\})$, by

$$(\boldsymbol{\pi}\boldsymbol{\phi})_\omega := \pi_{\tau^{-1}\omega}(\phi_{\tau^{-1}\omega}).$$

Then $\boldsymbol{\pi}$ admits a fixed point $\mathbf{h} = (h_\omega)_{\omega\in\Omega}$ in $L^\infty(\Omega, \mathcal{C} \cap \{\ell = 1\})$, and one has for almost every $\omega \in \Omega$, $\pi_\omega(h_\omega) = h_{\tau\omega}$.

Proof. Let $\mathcal{C}$, $\boldsymbol{\mathcal{L}}$ be as above. It is straightforward that

$$(\boldsymbol{\pi}^n\boldsymbol{\phi})_\omega = \pi^{(n)}_{\tau^{-n}\omega}\phi_{\tau^{-n}\omega} = \pi_{\tau^{-1}\omega} \circ \dots \circ \pi_{\tau^{-n}\omega}\phi_{\tau^{-n}\omega} = \frac{\mathcal{L}^{(n)}_{\tau^{-n}\omega}\phi_{\tau^{-n}\omega}}{\langle \ell, \mathcal{L}^{(n)}_{\tau^{-n}\omega}\phi_{\tau^{-n}\omega}\rangle} \tag{A.3.2}$$

where $\mathcal{L}^{(n)}_\omega = \mathcal{L}_{\tau^{n-1}\omega} \dots \mathcal{L}_\omega$ is the cocycle above (Ω, τ) generated by the random operator $\mathcal{L}_\omega$. From lemma A.2, one obtains easily (by induction) the following estimate on $\pi^{(n)}_\omega$: for $\omega \in \Omega$, and $x, y \in \mathcal{C}^*$,

$$\|\pi^{(n)}_{\tau^{-n}\omega}(x) - \pi^{(n)}_{\tau^{-n}\omega}(y)\|_E \le \frac{\Delta}{2K}\eta^{n-1} \tag{A.3.3}$$

where $\eta = \tanh\left(\frac{\Delta}{4}\right)$. It is noteworthy that the right hand side of the last estimate does not depends on ω, so that one has immediately

$$\|\boldsymbol{\pi}^n(\boldsymbol{\phi}) - \boldsymbol{\pi}^n(\boldsymbol{\psi})\|_{L^\infty(\Omega,E)} \le \frac{\Delta}{2K}\eta^{n-1} \tag{A.3.4}$$

for any $\phi, \psi \in L^\infty(\Omega, \mathcal{C} \cap \{\ell = 1\})$.

Take now a $\boldsymbol{h}_0 \in L^\infty(\Omega, \mathcal{C} \cap \{\ell = 1\})$, and define recursively $\boldsymbol{h}_{n+1} = \boldsymbol{\pi}(\boldsymbol{h}_n)$. By the previous estimate, the sequence $(\boldsymbol{h}_n)$ is Cauchy in $L^\infty(\Omega, E)$, and therefore it converges towards a $\boldsymbol{h} \in L^\infty(\Omega, E)$, which is the announced fixed point. $\square$

The result of this last theorem remains true for of regular Birkhoff cone, replacing the use of lemma A.6 by lemma A.2.

We continue this section with a formula giving a convenient representation of the characteristic exponent in terms of the fixed point $\boldsymbol{h}$ of $\boldsymbol{\pi}$.

Theorem A.9

Let $\mathcal{C}$ be a regular $\mathbb{C}$-cone, with associated linear form ℓ. Let $\boldsymbol{\mathcal{L}} \in L^\infty(\Omega, \mathcal{M}_{\mathcal{C}}(\Delta, \rho))$, a cocycle above (Ω, τ). Let $\boldsymbol{\pi}$ be the projection of $\boldsymbol{\mathcal{L}}$ on the affine hyperplan $\{\ell = 1\}$ (see 5.1.5), and $\boldsymbol{h}$ its fixed point.

Assume furthermore that $\mathbb{E}_\mu(\log \|\boldsymbol{\mathcal{L}}\|) < \infty$ (i.e $\log \|\boldsymbol{\mathcal{L}}\| \in L^1(\mu)$). Then one has

1. *The top characteristic exponent $\chi = \limsup \frac{1}{n} \log \|\mathcal{L}^{(n)}_\omega\|$ exists and is μ almost surely constant.*

2. *Furthermore, one has the following formula for χ:*

$$\chi = \int_\Omega \log |p_\omega| d\mu \tag{A.3.5}$$

where $p_\omega = \langle \ell, \mathcal{L}_{\tau^{-1}\omega} h_{\tau^{-1}\omega} \rangle$ for each $\omega \in \Omega$.

For the second statement, we will need the following lemma:

Lemma A.8
Let K the sectional aperture of $\mathcal{C}$, and for $\rho > 0$, let $\mathcal{C}[\rho] := \{x \in \mathcal{C},\ B(x, \rho\|x\|) \subset \mathcal{C}\}$. If $\rho > 0$ is sufficiently small, $\mathcal{C}[\rho]$ is not reduced to $\{0\}$.
Then for every $x \in \mathcal{C}[\rho]^*$, one has

$$\frac{1}{K} \left| \frac{\langle \ell, \mathcal{L}x \rangle}{\langle \ell, x \rangle} \right| \leq \|\mathcal{L}\|_E \leq \frac{K^2}{\rho} \left| \frac{\langle \ell, \mathcal{L}x \rangle}{\langle \ell, x \rangle} \right| \tag{A.3.6}$$

Let $y \in E$. Note that by assumption on $x \in \mathcal{C}[\rho]$, one has $x + ty \in \mathcal{C}^*$ for every $|t| \leq \rho\frac{\|x\|}{\|y\|}$. Therefore, $\mathcal{L}(x + ty) \in \mathcal{C}^*$ for such a t, and by the second estimate in lemma A.7, one has

$$\|\mathcal{L}y\| \leq \frac{K\|y\|}{\rho\|x\|} \|\mathcal{L}x\|$$

Together with the regularity condition $\|x\| \geq |\langle \ell, x \rangle| \frac{1}{K} \|x\|$ for $x \in \mathcal{C}^*$, we obtain

$$\frac{\|\mathcal{L}y\|}{\|y\|} \leq \frac{K^2}{\rho} \left| \frac{\langle \ell, \mathcal{L}x \rangle}{\langle \ell, x \rangle} \right|$$

and thus the second inequality is proven.

The first one simply comes from

$$|\langle \ell, \mathcal{L}x \rangle| \leq \|\mathcal{L}\|.\|x\| \leq K\|\mathcal{L}\|.|\langle \ell, x \rangle|$$

□

Proof. The first statement is an immediate consequence of Kingman theorem and the ergodicity of τ.

We can now prove formula (A.3.5).
Indeed, by construction of $\boldsymbol{h}$, one has almost surely for $\omega \in \Omega$

$$\mathcal{L}_\omega h_\omega = p_{\tau\omega} h_{\tau\omega} \tag{A.3.7}$$

Taking the product along the τ orbit, one obtains

$$\mathcal{L}_\omega^{(n)} h_\omega = \mathcal{L}_{\tau^{n-1}\omega} \dots \mathcal{L}_\omega h_\omega = \prod_{k=1}^{n} p_{\tau^k\omega} h_{\tau^n\omega} \tag{A.3.8}$$

so that

$$\frac{1}{n} \log |\langle \ell, \mathcal{L}_\omega^{(n)} h_\omega \rangle| = \frac{1}{n} \sum_{k=1}^{n} \log(|p_{\tau^k\omega}|) + \frac{1}{n} \log(|\langle \ell, h_{\tau^n\omega} \rangle|) \tag{A.3.9}$$

Note now that by lemma A.8, $\log(|\boldsymbol{p}|) \in L^1(\mu)$ if $\log(\|\boldsymbol{\mathcal{L}}\|) \in L^1(\mu)$. Thus, by Birkhoff ergodic theorem

$$\frac{1}{n}\sum_{k=1}^{n}\log(|p_{\tau^k\omega}|) \underset{n\to\infty}{\longrightarrow} \int_\Omega \log|p_\omega| d\mu(\omega)$$

It also follows from this lemma (with $\mathcal{L} = \mathcal{L}_\omega^{(n)}$ and $x = h_\omega$) that

$$\frac{1}{n}\log(\|\mathcal{L}_\omega^{(n)}\|) = \frac{1}{n}\log(|\langle \ell, \mathcal{L}_\omega^{(n)} h_\omega\rangle|) + \mathcal{O}(\frac{1}{n}) \tag{A.3.10}$$

As n tends to infinity, the left-hand term tends to χ, and the right-hand term to $\int_\Omega \log|p_\omega| d\mu(\omega)$ giving us the wanted identity. $\square$

Note that the same result remains true when $\mathcal{C}$ is assumed to be a regular Birkhoff cone. Indeed, the only thing left that might change is the validity of lemma A.8 and the second estimate in lemma A.7. Fortunately, lemma A.8 translates verbatim to the case of a regular Birkhoff cone (just by dropping the modulus !), and the other estimate has the following (simple) proof in the real case:

Indeed, if $\mathcal{C}$ is a regular Birkhoff cone, and there is $r > 0$ such that for every $x \in \mathcal{C}^*$, $y \in E$, $x + ty \in \mathcal{C}^*$ for $|t| < r$, then one has, taking $\ell \in E'$ of norm one given by the outer regularity of $\mathcal{C}$

$$\begin{cases} 0 < \langle \ell, x\rangle + t\langle \ell, y\rangle \\ 0 < \langle \ell, x\rangle - t\langle \ell, y\rangle \end{cases}$$

which gives

$$|t|\langle \ell, y\rangle \leq \langle \ell, x\rangle$$

so that $r\langle \ell, y\rangle \leq \langle \ell, x\rangle$. Thus, by the triangular inequality:

$$2r\|y\| \leq \|x + ry\| + \|x - ry\| \leq K[\langle \ell, x + ry\rangle + \langle \ell, x - ry\rangle] \leq 2K\langle \ell, x\rangle \leq 2K\|x\|$$

and thus $\|y\| \leq \frac{K}{r}\|x\|$.

Appendix B

Regularity estimates for composition operators on Hölder spaces

The main object of this section is to address the regularity problem for composition operators: $g \longmapsto [f \longmapsto f \circ g]$ in Hölder spaces. An important inspiration for the results presented here is a paper by de la Llave and Obaya, [15], particularly the following result:

Theorem B.1 ([15], Prop 6.2, (iii))
Let E,F,G be Banach spaces, and $\mathcal{U} \subset E$, $V \subset F$ open subsets. Let $k \geq 1$, $0 \leq \gamma < 1$ and $t = k + \gamma$. Let $s > t$ and $r \geq t$, and let $\mathcal{U} \subset C^r(\mathcal{U}, F)$. Then for every $g_1 \in \mathcal{U}$, there exists $\delta, \rho, M > 0$, such that for every $f \in C^s(V, G)$, every $g_2 \in C^r(\mathcal{U}, F)$ which verifies $\|g_1 - g_2\|_{C^r} \leq \delta$, one has $g_2 \in \mathcal{U}$, and

$$\|f \circ g_1 - f \circ g_2\|_{C^t} \leq M \|f\|_{C^s} \|g_1 - g_2\|^{\rho}_{C^r} \tag{B.0.1}$$

The estimates we establish in the following (lemmas B.1, B.2, B.4) are parameter variants of this theorem. They are used to prove lemma 4.1, which in turn is key for using theorem 3.1 to prove theorem 4.2.
In the first lemma, $g \longmapsto [f \longmapsto f \circ g]$ is Hölder continuous from $C^{1+\alpha}$ to $C^{1+\beta}$ with exponent $\gamma := \alpha - \beta$

B.1 Continuity estimates for the composition operator

Lemma B.1
Let $\mathcal{B}, E, F, G$ be Banach spaces, $\mathcal{U} \subset \mathcal{B}$, $V \subset E$, $W \subset F$ be open and bounded domains. Let $0 \leq \beta < \alpha < 1$, $\psi \in C^0(\mathcal{U} \times V, W)$ such that for every $u \in \mathcal{U}$, $\psi_u = \psi(u, .) \in C^{1+\alpha}(V, W)$, and every $x \in V$, $u \longmapsto \psi(u, x)$ is Lipschitz continuous, $u \longmapsto D_x \psi_u$ is α-Hölder.
Let f be such that for every $u \in \mathcal{U}$, $f_u \in C^{1+\alpha}(W, G)$ and the map $u \in \mathcal{U} \mapsto f_u \in C^1(W, G)$ is α-Hölder.

Let $u_0 \in \mathcal{U}$, and $h \in \mathcal{B}$ such that $u_0 + h \in \mathcal{U}$. Then $f_{u_0+h} \circ \psi_{u_0+h}$, $f_{u_0} \circ \psi_{u_0}$ are $C^{1+\beta}(V, G)$ maps, and we have

$$\|f_{u_0+h}\circ\psi_{u_0+h} - f_{u_0}\circ\psi_{u_0}\|_{C^{1+\beta}(V,G)} \leq [C_1\|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))} + C_2\|f\|_{C^\alpha(\mathcal{U},C^1(W,G))}]\|h\|_{\mathcal{B}}^\gamma \tag{B.1.1}$$

with C_1, C_2 given by (B.1.2),(B.1.3).

Proof. We introduce the following notations:

$$\begin{cases} L_{0,1} := \sup_{u\in\mathcal{U}} \|D_x\psi_u\|_\infty \quad L_{1,0} := \sup_{x\in V} \|D_u\psi_x\|_\infty \\ L_{0,1+\alpha} := \sup_{u\in\mathcal{U}} \|D_x\psi_u\|_{C^\alpha} \quad L_{\alpha,1} := \sup_{x\in V} \|D_x\psi_x\|_{C^\alpha} \end{cases}$$

For $u, v \in \mathcal{U}$ one can write:

$$\|f_u \circ \psi_u - f_v \circ \psi_v\|_{C^{1+\beta}} \leq \|(f_u - f_v)\circ\psi_u\|_{1+\beta} + \|f_v\circ\psi_u - f_v\circ\psi_v\|_{1+\beta}$$

For the first term, we want to estimate

$$\|(f_u - f_v)\circ\psi_u\|_{1+\beta} = \max\left(\|(f_u - f_v)\circ\psi_u\|_1, |D(f_u - f_v)\circ\psi_u.D\psi_u|_\beta\right)$$

It is easy to see that

$$\|(f_u - f_v)\circ\psi_u\|_{C^1} \leq \|D_x[f_u - f_v]\|_\infty\|D_x\psi_u\|_\infty \leq L_{0,1}|f|_{C^\alpha(\mathcal{U},C^1(W,G))}\|u - v\|^\alpha$$

Letting $x, x' \in V$, one has, when $d(x, x') \leq \|u - v\|_{\mathcal{B}}$

$$\begin{aligned}
&\|D_x[f_u](\psi(u,x)).D_x\psi(u,x) - D_x[f_u](\psi(u,x')).D_x\psi(u,x')\| \\
&\leq \|D_x[f](u,\psi(u,x')).[D_x\psi(u,x) - D_x\psi(u,x')]\| + \|[D_xf(u,\psi(u,x)) - D_xf(u,\psi(u,x'))].D_x\psi(u,x)\| \\
&\leq \left[|D_xf(u,.)|_{C^\alpha}\|D_x\psi_u\|_\infty^{1+\alpha} + \|D_xf_u\|_\infty|D_x\psi_u|_{C^\alpha}\right] d(x,x')^\beta\|u - v\|^\gamma \\
&\leq \|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}\left[L_{0,1}^{1+\alpha} + L_{0,1+\alpha}\right] d(x,x')^\beta\|u - v\|^\gamma
\end{aligned}$$

and similarly when $\|u - v\|_{\mathcal{B}} \leq d(x, x')$,

$$\begin{aligned}
\|D_xf(u,\psi(u,x)).D_x\psi(u,x) &- D_xf(v,\psi(u,x)).D_x\psi(u,x)\| \\
&\leq |D_xf(.,\psi(u,x))|_{C^\alpha}\|D_x\psi_u\|_\infty\|u - v\|^\gamma d(x,x')^\beta \\
&\leq \|f\|_{C^\alpha(\mathcal{U},C^1(W,G))}L_{0,1}\|u - v\|^\gamma d(x,x')^\beta
\end{aligned}$$

Thus we obtain the following bound:

$$\|(f_u - f_v)\circ\psi_u\|_{C^{1+\beta}} \leq \left[2\|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}(L_{0,1}^\alpha + L_{0,1+\alpha}) + \|f\|_{C^\alpha(\mathcal{U},C^1(W,G))}L_{0,1}(2 + \|u - v\|^\beta)\right]\|u - v\|^\gamma \tag{B.1.2}$$

The second term of the right hand can be treated as follows: we want to estimate

$$\|f_v\circ\psi_u - f_v\circ\psi_v\|_{C^{1+\beta}} = \max(\|f_v\circ\psi_u - f_v\circ\psi_v\|_{C^1}, \|D_x(f_v\circ\psi_u) - D_x(f_v\circ\psi_v)\|_{C^\beta})$$

.
For $x \in V$, one has :

$$\|D_x f(v,\psi(u,x)) \circ D_x\psi(u,x) - D_x f(v,\psi(v,x)) \circ D_x\psi(v,x)\|$$
$$\leq \|D_x f(v,\psi(u,x)) - D_x f(v,\psi(v,x))\|.\|D_x\psi(u,x)\| + \|D_x f(v,\psi(v,x))\|.\|D_x\psi(u,x) - D_x\psi(v,x)\|$$
$$\leq (L_{0,1}L^{\alpha}_{1,0} + L_{\alpha,1})\|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}\|u-v\|^{\alpha}.$$

For the Hölder semi-norm $|D_x(f_v \circ \psi_u) - D_x(f_v \circ \psi_v)|_{C^{\beta}}$, we have the following :
Let $x, x' \in V$, such that $\|d(x,x')\| \leq \|u-v\|$. Then

$$\|D_x f(v,\psi(u,x)) \circ D_x\psi(u,x) - D_x f(v,\psi(u,x')) \circ D_x\psi(u,x')\|$$
$$\leq \|D_x f(v,\psi(u,x)) - D_x f(v,\psi(u,x'))\|.\|D_x\psi(u,x)\| + \|D_x f(v,\psi(v,x))\|.\|D_x\psi(u,x) - D_x\psi(u,x')\|$$
$$\leq |D_x f(v,.)|_{C^{\alpha}} d(\psi(u,x)-\psi(u,x'))^{\alpha} + \|D_x f(v,.)\|_{\infty} L_{0,1+\alpha} d(x,x')^{\alpha}$$
$$\leq (L^{\alpha}_{0,1} + L_{0,1+\alpha})\|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))} d(x,x')^{\beta}\|u-v\|^{\alpha-\beta}$$

Similarly in the case $d(x,x') \geq \|u-v\|$, one has :

$$\|D_x f(v,\psi(u,x)) \circ D_x\psi(u,x) - D_x f(v,\psi(v,x)) \circ D_x\psi(v,x)\|$$
$$\leq \|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}(L^{\alpha}_{1,0}L_{0,1} + L_{\alpha,1})\|u-v\|^{\gamma} d(x,x')^{\beta}$$

Thus,

$$\|D_x(f_v \circ \psi_u) - D_x(f_v \circ \psi_v|)|_{C^{\beta}} \leq \|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}\left[(L^{\alpha}_{1,0}L_{0,1} + L_{\alpha,1})(2 + \|u-v\|^{\beta}) + L^{\alpha}_{0,1} + L_{0,1+\alpha}\right]\|u-v\|^{\gamma} \tag{B.1.3}$$

(B.1.1) readily follows. □

We now turn to the case $\gamma = 1$, i.e when considering the composition operator from $C^{1+\alpha}$ to C^{α}.

Lemma B.2
Let $\mathcal{B}$,E,F,G be Banach spaces, $\mathcal{U} \subset \mathcal{B}$, $V \subset E$, $W \subset F$ be open subsets.
Let $0 \leq \alpha < 1$ and $\psi \in C^{1+\alpha}(\mathcal{U} \times V, W)$, $f \in C^1(\mathcal{U}, C^{\alpha}(W,G))$ such that for every $u \in \mathcal{U}$, $f_u \in C^{1+\alpha}(W,G)$.
Then for every $u_0 \in \mathcal{U}$, and every $h \in \mathcal{B}$ such that $u_0+h \in \mathcal{U}$, the maps $f \circ \psi(u_0+h,.)$, $f \circ \psi(u_0,.)$ are α-Hölder and one has the estimate:

$$\|f(u_0+h) \circ \psi(u_0+h) - f(u_0) \circ \psi(u_0)\|_{C^{\alpha}} \leq \left[C_1\|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))} + C_2\|f\|_{C^1(\mathcal{U},C^{\alpha}(W,G))}\right]\|h\|_{\mathcal{B}} \tag{B.1.4}$$

with C_1, C_2 given by (B.1.10).

Proof. The case where f is a constant with respect to $u \in \mathcal{U}$ is a straightforward consequence of the mean value theorem. Taking the C^{α}-norm, one has for every $x \in V$.

$$\|f \circ \psi(u_0+h) - f \circ \psi(u_0)\|_{C^{\alpha}} \leq \|h\| \int_0^1 \|Df(\psi(u_0+th)) \circ D_u\psi(u_0+th)\|_{C^{\alpha}} dt \tag{B.1.5}$$

It is enough to establish the Lipschitz continuity that we wanted. Yet it is convenient to get a more precise estimate of $\|Df(\psi(u)) \circ D_u\psi(u)\|_{C^\alpha}$, for $u \in \mathcal{U}$.
Letting $x, x' \in W$, and taking the operator norm, one gets

$$\begin{aligned}
&\|Df(\psi(u,x)) \circ D_u\psi(u,x) - Df(\psi(u,x')) \circ D_u\psi(u,x')\| \\
&\qquad \leq \|Df(\psi(u,x)) - Df(\psi(u,x'))\|.\|D_u\psi(u,x)\| + \|Df(\psi(u,x'))\|.\|D_u\psi(u,x) - D_u\psi(u,x')\| \\
&\qquad \leq [|Df|_{C^\alpha}\|D_u\psi_u\|_{C^0}\|\psi_u\|_{C^1}^\alpha + \|Df\|_\infty\|D_u\psi_u\|_{C^\alpha}]d(x,x')^\alpha
\end{aligned}$$

so that

$$\|Df(\psi(u)) \circ D_u\psi(u)\|_{C^\alpha} \leq \|f\|_{C^{1+\alpha}}[\|D_u\psi_u\|_{C^0}\|D_x\psi_u\|_\infty^\alpha + \|D_u\psi_u\|_{C^\alpha}] \tag{B.1.6}$$

Turning to the general case, we also write, by virtue of the mean value theorem:

$$\|f_{u_0+h} \circ \psi_{u_0+h} - f_{u_0} \circ \psi_{u_0}\|_\alpha \leq \|h\| \int_0^1 \|D_uf(u_0+th, \psi(u_0+th)) + D_xf(u_0+th, \psi(u_0+th)).D_u\psi_(u_0+th)\|_\alpha d \tag{B.1.7}$$

$$\leq \|h\| \int_0^1 \|D_uf(u_0+th, \psi(u_0+th))\|_\alpha + \|D_xf(u_0+th, \psi(u_0+th)).D_u\psi(u_0+th)\|_\alpha dt \tag{B.1.8}$$

The second term in this last sum can be bounded by the same method as before. For the first term, we need to estimate $\|D_uf(u,\psi(u))\|_\alpha$. Using standard techniques, we can write

$$\|D_uf(u,\psi(u))\|_\alpha \leq \|D_uf(u,.)\|_\alpha\|D_x\psi(u,.)\|_\infty^\alpha dt \tag{B.1.9}$$

which gives (B.1.4), with the following explicit bound on the constants C_1, C_2:

$$C_1 \leq L_{1,0}L_{0,1}^\alpha + L_{1,\alpha} \tag{B.1.10}$$

$$C_2 \leq L_{0,1}^\alpha \tag{B.1.11}$$

□

Until now, we have treated the cases where $\beta \in [1+\alpha, 1]$, and the case $\beta = \alpha$. Using an **interpolation inequality**, it is possible to extend those results to the case $\beta \in (\alpha, 1)$.

Lemma B.3 (Same setting as before)
Let $f \in C^{1+\alpha}(\mathcal{U} \times W, G)$ and $\psi \in C^{1+\alpha}(\mathcal{U} \times V, W)$. Let $\beta \in (\alpha, 1)$, $u_0 \in \mathcal{U}$, $h \in \mathcal{B}$ such that $u_0 + h \in \mathcal{U}$.
Then the maps $f_{u_0+h} \circ \psi_{u_0+h}$ and $f_{u_0} \circ \psi_{u_0}$ are in $C^\beta(V,G)$ and one has

$$\|f_{u_0+h} \circ \psi_{u_0+h} - f_{u_0} \circ \psi_{u_0}\|_{C^\beta(V,G)} \leq C\|f\|_{C^{1+\alpha}(\mathcal{U}\times W,G)}\|h\|_{\mathcal{B}}^{1+\alpha-\beta} \tag{B.1.12}$$

with C an explicit constant depending on (B.1.2),(B.1.3), (B.1.10).

Proof. Let $\eta = \frac{1-\beta}{1-\alpha}$. As announced, we will use the following interpolation inequality for $\beta \in (\alpha, 1)$:

$$\|f_{u_0+h} \circ \psi_{u_0+h} - f_{u_0} \circ \psi_{u_0}\|_{C^\beta(V,G)} \leq C_\alpha \|f_{u_0+h} \circ \psi_{u_0+h} - f_{u_0} \circ \psi_{u_0}\|^{\eta}_{C^\alpha(V,G)} \|f_{u_0+h} \circ \psi_{u_0+h} - f_{u_0} \circ \psi_{u_0}\|^{1-\eta}_{C^1(V,G)} \tag{B.1.13}$$

$$\leq C_\alpha C_1^\eta C_2^{1-\eta} \|f\|^{\eta}_{C^{1+\alpha}(\mathcal{U}\times W,G)} \|f\|^{1-\eta}_{C^{1+\alpha}(\mathcal{U}\times W,G)} \|h\|^{\eta}_{\mathcal{B}} \|h\|^{(1-\eta)\alpha}_{\mathcal{B}} \tag{B.1.14}$$

$$\leq C'_\alpha \|f\|_{C^{1+\alpha}(\mathcal{U}\times W,G)} \|h\|^{\eta+(1-\eta)\alpha}_{\mathcal{B}} \tag{B.1.15}$$

where C_α, C_1, C_2 are explicitly computable constants (C_1, C_2 given by (B.1.2),(B.1.3), (B.1.10)), and we used that

$$\max\left(\|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}, \|f\|_{C^\alpha(\mathcal{U},C^1(W,G))}, \|f\|_{C^1(\mathcal{U},C^\alpha(W,G))}\right) \leq \|f\|_{C^{1+\alpha}(\mathcal{U}\times W,G)}$$

Noting that $\eta + \alpha(1-\eta) = 1+\alpha-\beta$, we get (B.1.12). □

We now turn to differentiability estimates:

B.2 Differentiability estimates for the composition operator

Lemma B.4
Let $\mathcal{B}$,E,F,G be Banach spaces, $\mathcal{U} \subset \mathcal{B}$, $V \subset E$, $W \subset F$ be open subsets.
Let $0 \leq \beta < \alpha < 1$ and $\psi \in C^{1+\alpha}(\mathcal{U} \times V, W)$, $f \in C^{1+\alpha}(\mathcal{U} \times W, G)$.

Let $u_0 \in \mathcal{U}$, and $h \in \mathbb{R}^d$ such that $u_0 + h \in \mathcal{U}$. Then $f \circ \psi(u_0)$, $f \circ \psi(u_0+h)$, $D_u(f\circ\psi)(u_0)$ are C^β maps, and we have

$$\|f(u_0+h)\circ\psi(u_0+h) - f(u_0)\circ\psi(u_0) - D_u(f\circ\psi)(u_0).h\|_{C^\beta} \leq C\|f\|_{C^{1+\alpha}(\mathcal{U}\times W,G)}\|h\|^{1+\gamma} \tag{B.2.1}$$

with C given by (B.2.4) *and* (B.2.7).

Proof. We introduce the following notations:

$$L_{0,1} := \sup_{u\in\mathcal{U}} \|D_x\psi(u,.)\|_\infty$$
$$L_{1,0} := \sup_{x\in V} \|D_u\psi(.,x)\|_\infty$$
$$L_{1,\alpha} := \sup_{u\in\mathcal{U}} \|D_u\psi(u,.)\|_{C^\alpha}$$
$$L_{1+\alpha,0} := \sup_{x\in V} \|D_u\psi(.,x)\|_\alpha$$

Using the mean value theorem and taking the norm, one can write :

$$\|f(u_0+h)\circ\psi(u_0+h) - f(u_0)\circ\psi(u_0) - D_u(f\circ\psi)(u_0).h\|_{C^\beta}$$

$$\leq \|h\|_{\mathcal{B}} \underbrace{\int_0^1 \|D_x f(u_0+th, \psi(u_0+th)).D_u\psi(u_0+th) - D_x f(u_0,\psi(u_0)).D_u\psi(u_0)\|_{C^\beta} dt}_{(I)} \tag{B.2.2}$$

$$+ \|h\|_{\mathcal{B}} \underbrace{\int_0^1 \|D_u f(u_0+th, \psi(u_0+th)) - D_u f(u_0,\psi(u_0))\|_\beta dt}_{(II)} \tag{B.2.3}$$

To estimate (I):=$\|D_x f(u_0+th,\psi(u_0+th))\circ D_u\psi(u_0+th) - D_x f(u_0,\psi(u_0))\circ D_u\psi(u_0)\|_{C^\beta}$, we apply the same method we used to establish (B.1.1).

Letting $x, x' \in V$, $u, v \in \mathcal{U}$, such that $\|u-v\| \le d(x,x')$ one obtains :

$$\frac{\|Df(u,\psi(u,x))\circ D_u\psi(u,x) - Df(v,\psi(v,x))\circ D_u\psi(v,x)\|}{d(x,x')^\beta}$$
$$\le \left[\|f\|_{C^1(W,C^\alpha(\mathcal{U},G))}\|D_u\psi(u)\|_\infty + \|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}(\|D_u\psi(u)\|_\infty^\alpha\|D_u\psi_x\|_\infty + |D_u\psi_x|_{C^\alpha})\right]\|u-v\|^\gamma$$

Similarly, in the case $d(x,x') < \|u-v\|$

$$\frac{\|Df(u,\psi(u,x))\circ D_u\psi(u,x) - Df(u,\psi(u,x'))\circ D_u\psi(u,x')\|}{d(x,x')^\beta} \le \|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}\left[\|D_x\psi(u)\|_\infty^\alpha\|D_u\psi\|_\infty + |D_u\psi(u)|_\alpha\right]\|u$$

which gives the following explicit bound on (I):

$$(I) \le \frac{2}{1+\gamma}\left[\|f\|_{C^1(W,C^\alpha(\mathcal{U},G))}L_{1,0} + \|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}(L_{1,0}^{1+\alpha} + L_{1,0}L_{0,1}^\alpha + L_{1+\alpha,0} + L_{1,\alpha})\right]\|h\|_{\mathcal{B}}^\gamma \tag{B.2.4}$$

We can also estimate (II) with our standard technique: in the case $d(x,x') \le \|h\|_{\mathcal{B}}$, one has:

$$\frac{\|D_u f(u_0+th,\psi(u_0+th,x)) - D_u f(u_0,\psi(u_0,x)) - D_u f(u_0+th,\psi(u_0+th,x')) + D_u f(u_0,\psi(u_0,x'))\|}{d(x,x')^\beta}$$
$$\le 2\sup_{u\in\mathcal{U}}|D_u f(u,.)|_\alpha \sup_{u\in\mathcal{U}}\|D_x\psi(u,.)\|_\infty^\alpha\|th\|^\gamma \tag{B.2.5}$$

and similarly in the case $d(x,x') > \|h\|_{\mathcal{B}}$:

$$\frac{\|D_u f(u_0+th,\psi(u_0+th,x)) - D_u f(u_0,\psi(u_0,x)) - (D_u f(u_0+th,\psi(u_0+th,x')) - D_u f(u_0,\psi(u_0,x')))\|}{d(x,x')^\beta}$$
$$\le 2\left[\sup_{u\in\mathcal{U}}|D_u f(u,.)|_\alpha \sup_{x\in V}\|D_u\psi(.,x)\|_\infty^\alpha + \sup_{y\in W}|D_u f(.,y)|_\alpha\right]t^\gamma\|h\|_{\mathcal{B}}^\gamma \tag{B.2.6}$$

This gives us the following bound on (II):

$$(II) \le \frac{2}{1+\gamma}\left[\|f\|_{C^1(\mathcal{U},C^\alpha(W,G))}(L_{0,1}^\alpha + L_{1,0}^\alpha) + \|f\|_{C^0(W,C^{1+\alpha}(\mathcal{U},G))}\right]\|h\|_{\mathcal{B}}^\gamma \tag{B.2.7}$$

Injecting estimates B.2.4 and B.2.7 in (B.2.2), one gets the following :

$$\|f(u_0+h)\circ\psi(u_0+h) - f(u_0)\circ\psi(u_0) - D_u(f\circ\psi)(u_0).h\|_{C^\beta}$$
$$\le \left[C_1\|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))} + C_2\|f\|_{C^1(W,C^\alpha(\mathcal{U},G))} + C_3\|f\|_{C^1(\mathcal{U},C^\alpha(W,G))} + C_4\|f\|_{C^0(W,C^{1+\alpha}(\mathcal{U},G))}\right]\|h\|^{1+\gamma}$$

which gives the promised result with C_1, C_2, C_3, C_4 given by the right-hand side on B.2.4 and B.2.7. □

At order 2, the mechanism is the same, although the computation are heavier:

Lemma B.5 (Same setting as in the previous cases)
Let $\psi \in C^{2+\alpha}(\mathcal{U} \times V, W)$ and $f \in C^{2+\alpha}(\mathcal{U} \times W, G)$.

Then the maps $f_{u_0+h} \circ \psi_{u_0+h}$, $f_{u_0} \circ \psi_{u_0}$, $D_u[f \circ \psi](u_0)$, $D_u^2[f \circ \psi](u_0)$ are in $C^\beta(V, G)$ and one has the following quantitative estimate:

$$\|f_{u_0+h} \circ \psi_{u_0+h} - f_{u_0} \circ \psi_{u_0} - D_u[f \circ \psi](u_0).h - D_u^2[f \circ \psi](u_0)(h,h)\|_{C^\beta} \leq C\|f\|_{C^{2+\alpha}(\mathcal{U}\times W,G)}\|h\|^{2+\gamma} \tag{B.2.8}$$

with C a constant, given by (B.2.12),(B.2.13),(B.2.14).

Proof. We introduce the following notations:

$$L_{2,0} := \sup_{u\in\mathcal{U}} \sup_{x\in V} \|D_u^2\psi(u,x)\|$$

$$L_{2,\alpha} := \sup_{u\in\mathcal{U}} \|D_u^2\psi(u,.)\|_{C^\alpha}$$

$$L_{2+\alpha,0} := \sup_{x\in V} \|D_u^2\psi(,.x)\|_{C^\alpha}$$

As usual, we use the mean-value theorem (this time at order 2) and take the C^β norm : this yields

$$\begin{aligned} &\|f_{u_0+h} \circ \psi_{u_0+h} - f_{u_0} \circ \psi_{u_0} - D_u[f \circ \psi](u_0).h - D_u^2[f \circ \psi](u_0)(h,h)\|_{C^\beta} \\ &\qquad \leq \|h\|_{\mathcal{B}}^2 \int_0^1 \|D_u^2[f \circ \psi](u_0+th) - D_u^2[f \circ \psi](u_0)\|_{C^\beta} dt \end{aligned} \tag{B.2.9}$$

The second differential with respect to u of $f \circ \psi$ is given by a classical computation:

$$D_u^2[f \circ \psi](u) = D_u^2 f(u,\psi_u) + 2D_xD_u f(u,\psi_u)[D_u\psi(u),.] + D_x^2 f(u,\psi_u)[D_u\psi_u, D_u\psi_u] + D_x f(u,\psi_u).D_u^2\psi(u) \tag{B.2.10}$$

which entails that

$$\begin{aligned} &\int_0^1 \|D_u^2[f \circ \psi](u_0+th) - D_u^2[f \circ \psi](u_0)\|_{C^\beta} dt \\ &\leq \int_0^1 \|D_u^2 f(u_0+th,\psi(u_0+th)) - D_u^2 f(u_0,\psi(u_0))\|_{C^\beta} dt := (I) \\ &+ 2\int_0^1 \|D_xD_u f(u_0+th,\psi(u_0+th)).[D_u\psi(u_0+th),.] - D_xD_u f(u_0,\psi(u_0)).[D_u\psi(u_0),.]\|_{C^\beta} dt := (II) \\ &+ \int_0^1 \|D_x^2 f(u_0+th,\psi(u_0+th)).[D_u\psi(u_0+th), D_u\psi(u_0+th)] - D_x^2 f(u_0,\psi(u_0)).[D_u\psi(u_0), D_u\psi(u_0)]\|_{C^\beta} dt := (III) \\ &+ \int_0^1 \|D_x f(u_0+th,\psi(u_0+th)).D_u^2\psi(u_0+th) - D_x f(u_0,\psi(u_0)).D_u^2\psi(u_0)\|_{C^\beta} dt := (IV) \end{aligned} \tag{B.2.11}$$

One thus needs to bound (I),(II),(III) and (IV). We once again use our standard trick, and we will skip the details as they are very similar to our previous computations.
For (I), one obtains

$$(I) \leq \frac{2\|h\|_{\mathcal{B}}^\gamma}{1+\gamma}\left[\|f\|_{C^2(\mathcal{U},C^\alpha(W,G))}(L_{0,1}^\alpha + L_{1,0}^\alpha) + \|f\|_{C^0(W,C^{2+\alpha}(\mathcal{U},G))}\right] \tag{B.2.12}$$

For (II), one obtains

$$(II) \leq \frac{2\|h\|_{\mathcal{B}}^{\gamma}}{1+\gamma}\left[\|f\|_{C^1(\mathcal{U},C^{1+\alpha}(W,G))}(L_{1,0}L_{0,1}^{\alpha}+L_{1,0}^{1+\alpha}+L_{1,\alpha}+L_{1+\alpha,0})+\|f\|_{C^1(W,C^{1+\alpha}(\mathcal{U},G))}L_{1,0}\right] \tag{B.2.13}$$

and similarly for (III) and (IV):

$$(III) \leq \frac{2\|h\|^{\gamma}}{1+\gamma}\left[\|f\|_{C^0(\mathcal{U},C^{2+\alpha}(W,G))}(L_{1,0}^2(L_{0,1}^{\alpha}+L_{1,0}^{\alpha})+2L_{1,0}(L_{1,\alpha}+L_{1+\alpha,0}))+\|f\|_{C^2(W,C^{\alpha}(\mathcal{U},G))}L_{1,0}^2\right] \tag{B.2.14}$$

$$(IV) \leq \frac{2\|h\|^{\gamma}}{1+\gamma}\left[\|f\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}(L_{2,0}(L_{1,0}^{\alpha}+L_{0,1}^{\alpha})+L_{2,\alpha}+L_{2+\alpha,0})+\|f\|_{C^1(W,C^{\alpha}(\mathcal{U},G))}L_{2,0}\right] \tag{B.2.15}$$

which conclude the proof. □

Remark B.1

- *Of particular interest to us will be the case where $f(u,x)=g(u,x)\phi(x)$ Indeed, this is the kind of expression one has to study when considering transfer operators. In this case, one obtains that the composition operator W_u, defined by*

$$W_u(\phi) := (g_u\phi)\circ\psi_u$$

  is C^{r-s} when seen as an operator from C^r to C^s, with $s<r$.

- *It is possible to lead a similar analysis at any order of differentiability (provided the regularity of f,ψ is big enough). However, as we just saw, it involves very heavy computation, particularly to derive explicit bounds on the constants at play.*
  Those explicit bounds are of particular interest to extend lemmas B.1, B.2 and B.4 to a ***random*** *context.*

B.3 Regularity estimates for the random composition operator

We now consider the case where the maps we compose are chosen at random, according to some probability law $\mathbb{P}$. The crucial assumption here is the boundedness w.r.t the random parameter; in that context, our previous results extends straightforwardly.

Lemma B.6

Let (Ω,μ) be a probability space, $\mathcal{B},E,F,G$ be Banach spaces, $\mathcal{U}\subset\mathcal{B}$, $V\subset E$, $W\subset F$ be open domains.
Let $0\leq\beta<\alpha<1$, $\gamma:=\alpha-\beta>0$ $\psi\in L^{\infty}(\Omega,Lip(\mathcal{U}\times V,W))$. Furthermore, we assume that for every $u\in\mathcal{U}$, $\psi_{\omega,u}=\psi(\omega,u,.)\in C^{1+\alpha}(V,W)$ and thatu $\longmapsto D_x\psi_{\omega,u}$ is α-Hölder

Let $f\in L^{\infty}(\Omega,C^{1+\alpha}(\mathcal{U}\times W,G))$, let $u_0\in\mathcal{U}$, and $h\in\mathcal{B}$ such that $u_0+h\in\mathcal{U}$. Then $f(\omega,u_0+h)\circ\psi(\omega,u_0+h)$, $f(\omega,u_0)\circ\psi(\omega,u_0)\in L^{\infty}(\Omega,C^{1+\beta}(V,G))$, and we have

$$\begin{aligned}\|f(u_0+h)\circ\psi(u_0+h)-f(u_0)\circ\psi(u_0)\|_{L^{\infty}(\Omega,C^{1+\beta}(V,G))}\\ \leq [C_1\|f\|_{L^{\infty}(\Omega,C^0(\mathcal{U},C^{1+\alpha}(W,G)))}+C_2\|f\|_{L^{\infty}(\Omega,C^{\alpha}(\mathcal{U},C^1(W,G)))}]\|h\|^{\gamma}\end{aligned} \tag{B.3.1}$$

with C_1,C_2 given by taking an essential bound in (B.1.2),(B.1.3).

Proof. By virtue of lemma B.1, one can write, for each $\omega \in \Omega$,

$$\|f_{\omega,u_0+h}\circ\psi_{\omega,u_0+h}-f_{\omega,u_0}\circ\psi_{\omega,u_0}\|_{C^{1+\beta}} \leq [C_1(\omega)\|f_\omega\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))}+C_2(\omega)\|f_\omega\|_{C^\alpha(\mathcal{U},C^1(W,G))}]\|h\|_{\mathcal{B}}^\gamma \tag{B.3.2}$$

with $C_1(\omega), C_2(\omega)$ given by (B.1.2),(B.1.3).
Our assumptions insures us that the right hand term in (B.3.2) admits an (essential) upper bound. Taking the essential supremum, (B.3.1) follows straightforwardly. ☐

It is noteworthy that one should now take the following definitions for the quantities appearing in constants C_1, C_2:

$$L_{0,1} = \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{u\in\mathcal{U}} \|D_x\psi_{\omega,u}\|_\infty < \infty, \; L_{1,0} = \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in V} \|D_u\psi_{\omega,x}\|_\infty < \infty$$

$$L_{0,1+\alpha} := \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{u\in\mathcal{U}} |D_x\psi_{\omega,u}|_\alpha < \infty, \; L_{\alpha,1} := \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in V} |D_x\psi_{\omega,x}|_\alpha < \infty$$

Lemma B.7
Let (Ω,μ) be a probability space, $\mathcal{B}, E, F, G$ be Banach spaces, $\mathcal{U} \subset \mathcal{B}$, $V \subset E$, $W \subset F$ be open domains.
Let $0 < \alpha < 1$, $f \in L^\infty(\Omega, C^{1+\alpha}(\mathcal{U}\times W, G))$, and $\psi \in L^\infty(\Omega, C^{1+\alpha}(\mathcal{U}\times V, W))$. For $u_0 \in \mathcal{U}$, and $h \in \mathcal{B}$ such that $u_0+h \in \mathcal{U}$, one has that $f(u_0+h)\circ\psi(u_0+h)$, $f(u_0)\circ\psi(u_0) \in L^\infty(\Omega, C^\alpha(V,G))$ and

$$\|f(u_0+h)\circ\psi(u_0+h)-f(u_0)\circ\psi(u_0)\|_{L^\infty(\Omega,C^\alpha)} \leq \left[C_1\|f\|_{L^\infty(C^0(\mathcal{U},C^{1+\alpha}(W,G)))} + C_2\|f\|_{L^\infty(C^1(\mathcal{U},C^\alpha(W,G)))}\right]\|h\|_{\mathcal{B}} \tag{B.3.3}$$

with C_1, C_2 given by (B.1.10).

Proof. Once again we write, for a fixed $\omega \in \Omega$,(B.1.4):

$$\|f(\omega,u_0+h)\circ\psi(\omega,u_0+h)-f(\omega,u_0)\circ\psi(\omega,u_0)\|_{C^\alpha} \leq \left[C_1(\omega)\|f_\omega\|_{C^0(\mathcal{U},C^{1+\alpha}(W,G))} + C_2(\omega)\|f_\omega\|_{C^1(\mathcal{U},C^\alpha(W,G))}\right]\|h\|_{\mathcal{B}} \tag{B.3.4}$$

where the (random) constants $C_1(\omega)$ and $C_2(\omega)$ are given by (B.1.10). The right-hand term in (B.3.4) admits an essential upper-bound, and thus so does the left-hand side. (B.3.3) immediately follows. ☐

The same generalizations hold for our differentiability estimates:

Lemma B.8 (Same setting as before)
Let $\psi \in L^\infty(\Omega, C^{1+\alpha}(\mathcal{U}\times V, W))$ and $f \in L^\infty(\Omega, C^{1+\alpha}(\mathcal{U}\times W, G))$. Let $u_0 \in \mathcal{U}$, $h \in \mathcal{B}$ such that $u_0+h \in \mathcal{U}$.
Then the random maps $f_{\omega,u_0+h}\circ\psi_{\omega,u_0+h}, f_{\omega,u_0}\circ\psi(\omega,u_0), D_u[f_\omega\circ\psi_\omega](u_0)$ are $L^\infty(\Omega, C^\beta(V,G))$ and one has

$$\|f_{u_0+h}\circ\psi_{u_0+h} - f_{u_0}\circ\psi_{u_0} - D_u[f\circ\psi](u_0).h\|_{L^\infty(\Omega,C^\beta(V,G))} \leq C\|f\|_{L^\infty(\Omega,C^{1+\alpha}(\mathcal{U}\times W,G))}\|h\|_{\mathcal{B}}^{1+\gamma} \tag{B.3.5}$$

with C given by taking an essential bound in (B.2.1).

Lemma B.9 (Same setting as before)
Let $\psi \in L^\infty(\Omega, C^{2+\alpha}(\mathcal{U} \times V, W))$ *and* $f \in L^\infty(\Omega, C^{2+\alpha}(\mathcal{U} \times W, G))$*. Let* $u_0 \in \mathcal{U}$*,* $h \in \mathcal{B}$ *such that* $u_0 + h \in \mathcal{U}$*.*
Then the random maps $f_{\omega,u_0+h} \circ \psi_{\omega,u_0+h}, f_{\omega,u_0} \circ \psi(\omega, u_0), D_u[f_\omega \circ \psi_\omega](u_0), D_u^2[f_\omega \circ \psi_\omega](u_0)$ *are* $L^\infty(\Omega, C^\beta(V, G))$ *and one has*

$$\|f_{u_0+h}\circ\psi_{u_0+h}-f_{u_0}\circ\psi_{u_0}-D_u[f\circ\psi](u_0).h-D_u^2[f\circ\psi](u_0)[h,h]\|_{L^\infty(\Omega,C^\beta(V,G))} \leq C\|f\|_{L^\infty(\Omega,C^{2+\alpha}(\mathcal{U}\times W,G))}\|h\|_{\mathcal{B}}^{2+\gamma} \tag{B.3.6}$$

with C *given by taking an essential bound in* (B.2.8)*.*

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Titre : Étude de systèmes dynamiques avec perte de régularité

Mots Clefs : Systèmes uniformément dilatants, systèmes dynamiques aléatoires, fonctions implicites, réponse linéaire

Résumé : L'objet de cette thèse est le développement d'un cadre unifié pour étudier la régularité de certains éléments caractéristiques des dynamiques chaotiques (pression/entropie topologique, mesure de Gibbs, exposants de Lyapunov) par rapport à la dynamique elle même. Le principal problème technique est la perte de régularité venant de l'utilisation d'un opérateur de composition, l'opérateur de transfert, dont les propriétés spectrales sont intimement liées aux "éléments caractéristiques" ci-dessus. Pour surmonter ce problème, nous établissons un théorème de régularité par rapport aux paramètres pour des points fixes, dans un esprit proche du théorème des fonctions implicites de Nash Moser. Nous appliquons ensuite cette approche "point fixe" au problème de la réponse linéaire (régularité de la mesure invariante du système par rapport aux paramètres) pour une famille de dynamiques uniformément dilatantes.
Dans un second temps, nous étudions la régularité du plus grand exposant de Lyapunov d'un produit aléatoire d'applications dilatantes, s'appuyant sur notre théorème de régularité et la théorie des contractions de cônes. Nous en déduisons la régularité par rapport aux paramètres de la mesure stationnaire, de la variance dans le théorème limite central, et d'autres quantités dynamiques d'intérêt.

Title : On loss of regularity in dynamical systems

Keys words : Uniformly expanding systems, Random dynamical systems, Implicit functions, Linear response

Abstract : The aim of this thesis is the development of a unified framework to study the regularity of certain characteristics elements of chaotic dynamics (Topological presure/entropy, Gibbs measure, Lyapunov exponents) with respect to the dynamic itself. The main technical issue is the regularity loss occuring from the use of a composition operator, the transfer operator, whose spectral properties are intimately connected to the aformentionned "characteristics elements". To overcome this issue, we developped a regularity theorem for fixed points (with respect to parameter), in the spirit of the implicit function theorem of Nash and Moser. We then apply this "fixed point" approach to the linear response problem (studying the regularity of the system invariant measure w.r.t parameters) for a family of uniformly expanding maps.
In a second time, we study the regularity of the top characteristic exponent of a random prduct of expanding maps, building from our regularity theorem and cone contraction theory. We deduce from this regularity with respect to parameters for the stationnary measure, the variance in the central limit theorem, and other quantities of dynamical interest.